Probability III – 2009/10

Exercise Sheet 6 **JRJ**

Please hand in your answer to Question 3 only to the red box on the second floor of the maths building by 10am on Thursday 4 March.

You are strongly encouraged to attempt all the questions on this sheet.

If you want feedback on your answers to any of the questions then see me in a class or office hour.

1. Consider the random walk on $S = \{0, 1, 2, \ldots, n\}$ with reflecting boundary and $p = 1/2$. That is $p_{i,i+1} = p_{i,i-1} = 1/2$ for $1 \leq i \leq n-1$, $p_{0,1} = p_{n,n-1} = 1$.

   a) Find the equilibrium distribution for this process.

   b) Suppose that $X_0 = 0$ what is the expected time of the first return to 0.

   c) Once the chain has been run for a long time what is the expected proportion of time spent in state 1.

   d) Once the chain has been run for a long time what is the expected proportion of time for which the process is in state 0,1 or 2.

2. A gambler wagers £1 on a fair game (equal probability of winning and losing). If he wins he leaves the casino; if he loses then he wagers £2 on a fair game. This process continues as follows. If he wins the nth game he leaves the casino; if he loses the nth game he wagers £2^n in the $(n+1)$st game.

   a) What is the probability that the gambler makes exactly t wagers?

   b) What is the probability that he eventually leaves the casino?

   c) What can you say about his overall gain or loss when he leaves the casino.

   d) What is the expectation of the number of wagers made?

   e) What is the expectation of the number of pounds wagered in the final game?

   f) Does the difference between your ansewrs to d) and e) surprise you? What do you think about the gambler's strategy?

3.

a) Let K_5 be the graph with vertices $\{1,2,3,4,5\}$ and edges $\{12,13,14,15,23,24,25,34,35,45\}$ (so every pair of vertices are joined). Consider the random walk on K_5.

   i) How do you know that this has a unique equilibrium distribution?

   ii) How do you know that it has a limiting distribution?

   iii) Without solving any equations write down the equilibrium distribution.

   iv) Find the transition matrix P and calculate P^2 and P^4.

b) Let C_5 be the graph with vertices $\{1,2,3,4,5\}$ and edges $\{12,23,34,45,15\}$ (a cycle). Repeat parts i)-iv) above for the graph C_5.

c) Comment on which of the two random walks appears to be converging faster to its limiting distribution.

4. For which graph on n vertices is the expected return time to vertex 1 as small as it can be?

For which graph on n vertices is the expected return time to vertex 1 as large as it can be?