Probability III – 2009/10

Exercise Sheet 5 **JRJ**

Please hand in your answer to Question 4 only to the red box on the second floor of the maths building by 10am on Thursday 18 February.

You are strongly encouraged to attempt all the questions on this sheet. In particular, Question 3 will help with Question 4.

If you want feedback on your answers to any of the questions then see me in a class or office hour.

The fact that

$$\frac{n^n\sqrt{2\pi n}}{e^n} \le n! \le \frac{11}{10}\frac{n^n\sqrt{2\pi n}}{e^n}.$$

or another version of Stirling's approximation (which may be familiar to some of you) will be of use.

1. Consider the Markov chain on state space $S = \{1,2,3\}$ with transition matrix

$$\begin{pmatrix} 0 & 1/5 & 4/5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

For each state $s \in S$ calculate directly the distribution of R_s (the time of first return to s) and $\mathbb{E}(R_s)$. Use this to write down the equilibrium distribution for the chain. Check that the answer you get is the same as that obtained by the method of Question 1d on Sheet 3.

2. Consider the Markov chain on state space $S = \{0,1,2,\dots\}$ with transition probabilities $p_{i,i+1} = \frac{i+1}{i+2}$, $p_{i,0} = \frac{1}{i+2}$ for $i \ge 0$ (all others being 0). Calculate $f_{00}^{(t)}$ and f_{00} for this chain and deduce from this that the chain is recurrent. Calculate $\mathbb{E}(R_0)$ and decide whether the chain is positive recurrent or null recurrent.

Can you see how this chain is a variant of the success-runs chain?

3. Consider the random walk on $\mathbb{Z}$ in which the transition probabilities are $p_{i,i+2} = p_{i,i-1} = \frac{1}{2}$ for all i (that is we are equally likely to move distance 2 to the left or distance 1 to the right).

a) Explain why the number of steps in any path from 0 to 0 is a multiple of 3.

b) Show that if t is a multiple of 3 there are $\binom{t}{\frac{t}{3}}$ different t-step paths from 0 to 0.

c) Deduce that

$$p_{00}^{(t)} = \begin{cases} 2^{-t}\binom{t}{\frac{t}{3}} & \text{if } t \text{ is a multiple of 3} \\ 0 & \text{otherwise} \end{cases}$$

d) By considering $\sum_{t\ge1} p_{00}^{(t)}$ decide whether the chain is recurrent or transient.

4.

a) Consider the random walk on $\mathbb{Z}$ with transition probabilities $p_{i,i+1} = p_{i,i-1} = 1/2$ for all i.

i) Find an expression for $p_{00}^{(t)}$ for all t.

ii) Use your answer to part i) to calculate $f_{00}^{(t)}$ for $t \leq 6$.

iii) Is state 0 positive recurrent, null recurrent, or transient?

iv) Is state 1526 positive recurrent, null recurrent, or transient?

b) Consider the random walk on $\mathbb{Z}$ with transition probabilities $p_{i,i+1} = 1/3$, $p_{i,i-1} = 2/3$ for all i.

i) Find an expression for $p_{00}^{(t)}$ for all t.

ii) Use your answer to part i) to calculate $f_{00}^{(t)}$ for $t \leq 6$.

iii) Is state 0 positive recurrent, null recurrent, or transient?

iv) Is state 1526 positive recurrent, null recurrent, or transient?

5. Prove that null recurrence is a class property (that is if a is a null recurrent state and b is in the same communicating class as a then b is also null recurrent).