Probability III – 2009/10

Exercise Sheet 4 **JRJ**

Please hand in your answers to Questions 3 and 4 only to the red box on the second floor of the maths building by 10am on Thursday 11 February.

You are strongly encouraged to attempt all the questions on this sheet.

If you want feedback on any of the questions then see me in a class or office hour.

1. Look back to your answer to Question 1 on Sheet 3. For which of the 5 parts do we have that the expected proportion of time spent in a particular state up to time t tends to a limit independently of the initial distribution? In each case say what the expected proportion of time spent in state 1 is in the long run, and whether it depends on the initial distribution.

2. Consider the Markov chain on state space $\{1,2,3,4,5,6,7,8,9\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 1/6 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 1/4 & 1/4 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 9/10 & 0 & 0 & 0 & 0 & 0 & 1/10 \\ 1/2 & 1/3 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/7 & 6/7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & 1/4 \end{pmatrix}.$$

Find the communicating classes of states of the chain. For each state say whether it is recurrent of transient, giving a brief explanation of why (you do not need to calculate the first return probabilities exactly to do this).

3. Consider the Markov chain on state space $\{1,2,3,4,5\}$ with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 4/5 & 1/5 \\ 0 & 2/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

a) Calculate the first return probability $f_{11}^{(t)}$ and the return probability f_{11}.

b) Why would it become harder to do this if state 2 was modified so that p_{21}, p_{22}, p_{23} were all positive?

c) How could you calculate f_{11} without working out the $f_{11}^{(t)}$?

4. If a is a state of a Markov chain, we say that a is 2-periodic if $p_{aa}^{(k)} = 0$ for all odd k.

a) Show that if i is 2-periodic and j is a element of the same communicating class as i then j is 2-periodic. (We say that 2-periodicity is a class property.)

   [Hint: Assume that i is 2-periodic but j is not 2-periodic and deduce a contradiction.]

b) Can you find a Markov chain which contains both a loop (that is a state s with $p_{ss} > 0$) and a 2-periodic state.

c) Can you find an irreducible Markov chain which contains both a loop and a 2-periodic state?

5. (For those who took Introduction to Algebra.) Show that $\leftrightarrow$ is an equivalence relation on S.