Probability III – 2009/10

Exercise Sheet 3 **JRJ**

Please hand in your answer to Question 2 only to the red box on the second floor of the maths building by 10am on Thursday 4 February.

You are strongly encouraged to attempt all the questions on this sheet. Question 1 is the most important to do to make sure you understand the basics of this part of the course.

If you want feedback on any of the questions then see me in a class or office hour.

1. For each of the following transition matrices say whether the corresponding Markov chain is irreducible, regular, or neither. Which of them have a unique equilibrium distribution? Which of them have a limiting distribution?

a)
$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 4/5 & 0 & 0 & 1/5 \end{pmatrix}$$

b)
$$\begin{pmatrix} 2/5 & 1/5 & 2/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c)
$$\begin{pmatrix} 2/5 & 1/5 & 2/5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

d)
$$\begin{pmatrix} 0 & 1/5 & 4/5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

e)
$$\begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$$

Where $a+b+c+d = 1$ and $0 < a,b,c,d < 1$.

2.

a) i) Prove that if an irreducible Markov chain on a finite state space has a state i with $p_{ii} > 0$ then it is regular.

ii) Does the same result hold if we allow the state space to be infinite? Justify your answer

b) Let $X_0, X_1, \ldots$ be the Markov chain with state space $\{1,2,3,4\}$ and transition matrix

$$\begin{pmatrix} p & 0 & 1/2-p & 1/2 \\ 0 & 0 & 1/3 & 2/3 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

for some $0 \leq p \leq 1/2$.

i) For which values of p is this Markov chain regular?

ii) Find the limiting distribution for this Markov chain when $p = 1/10$.

iii) Suppose that the X_i forming the Markov chain of part ii) represent the location on day i of a study of a gorilla moving randomly between 4 regions of a forest. Write one sentence which explains the significance of the first coordinate in the limiting distribution. Your answer should make sense to an intelligent non-mathematician and should not use any special mathematical notation or terminology.

3. Show that if a Markov chain has a transition matrix in which all column sums are equal to 1 then the uniform distribution is an equilibrium distribution.

4. Look back at your answer to Question 4 on Sheet 1. Show that the chain you describe in part a) of that question is regular. and find its limiting distribution. What can you say about how quickly the distribution converges to the limiting distribution?

5. Consider a maze of n rooms with passages joining some of them. Define a random walk in the maze by starting in room 1 and at each step choosing a passage out of your current room uniformly at random. Let X_i be your room number after i steps. Find an equilibrium distribution for this Markov chain. Under what conditions is the chain irreducible? Under what conditions in the chain regular? (The last part is harder, if you can't manage it at least try to find some examples which are regular and some examples which are not and try to formulate a plausible guess at the answer.)