Probability III – 2009/10

Exercise Sheet 2 JRJ

Please hand in your answer to Question 4 only to the red box on the second floor of the maths building by 10am on Thursday 28 January.

You are strongly encouraged to attempt all the questions on this sheet. In particular I advise you to do question 2 before looking at question 4. If you want feedback on any other questions then see me in a class or office hour.

1. Let $X_0, X_1, \dots$ be the Markov chain on state space $\{1,2\}$ with transition matrix

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}.$$

.

   a) By diagonalising P find P^5.

   b) Find $\mathbb{P}(X_5 = 1 | X_0 = 1)$ and $\mathbb{P}(X_5 = 1 | X_0 = 2)$.

   c) Find $\mathbb{P}(X_{100} = 1 | X_0 = 1)$ and $\mathbb{P}(X_{100} = 1 | X_0 = 2)$.

   d) Comment on your answers to b) and c).

2. A Markov chain on state space $\{1,2,3,4\}$ has transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   a) Which states are absorbing?

   b) Find the probability that the process ends up in state 1 given that it starts in state 2?

3. I have £1 and you have £2. We play the following game. A coin which has probability p of showing heads is tossed. If the coin shows heads then you pay me £1; if it shows tails then I pay you £1. The game stops when one of us has no money left.

   a) Describe a Markov chain which models this process.

   b) For what value of p is the game fair in the sense that the probability that I end up with no money is equal to the probability that you end up with no money?

4. A Markov chain on state space $\{1,2,3,4,5\}$ has transition matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4/5 & 1/5 & 0 \\ 0 & 1/6 & 2/3 & 0 & 1/6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The process starts in state 1.

a) Which states are absorbing?

b) Calculate the probability that the process is absorbed at state 4 (ends up at state 4).

c) Calculate the expectation of the time of absorption.

d) Calculate the expectation of the number of visits to state 2 before absorption.

e) Suppose that you lose £5 for each visit to state 2 and gain £10 for each visit to state 3. Use first step analysis to calculate the expectation of the amount you gain.

f) How could you work out the answer to e) from c) and d) without doing the first step analysis of part e)?

5. A standard die is rolled repeatedly until the sum of two consecutive rolls is exactly 4.

a) Show how to model this process using a Markov chain with 36 states.

b) Show how to model this process using a Markov chain with a significantly smaller number of states.

c) Calculate the expectation of the number of rolls the process lasts.

6. Find a simple example of a Markov chain on a finite state space with two absorbing states for which the probability that the process eventually reaches an absorbing state is strictly between 0 and 1. Do the first step analysis equations for finding the probability of absorption in a particular absorbing state in your chain have a unique solution? If not how would you identify the correct solution in your example?