Probability III – 2009/10

Exercise Sheet 1 **JRJ**

Please hand in your answer to Question 1 only to the red box on the second floor of the maths building by 10am on Thursday 21 January.

If you want feedback on any other questions then see me in a class or office hour.

1. A standard 6-sided die is rolled repeatedly. Let X_i be the number of 6s seen in the first i rolls. Let Y_i be the largest number seen in the first i rolls.

a) Explain briefly why $X_1, X_2, X_3, \ldots$ is a Markov chain and give the transition probabilities and the transition graph.

b) Explain briefly why $Y_1, Y_2, Y_3, \ldots$ is a Markov chain and give the transition matrix.

The die is modified so that it behaves normally on the first roll but threafter it can never produce the same number twice in succession (with the other 5 possibilities being equally likely). The random variables X_i and Y_i are defined as before.

c) Is $X_1, X_2, X_3, \ldots$ a Markov chain?

d) Is $Y_1, Y_2, Y_3 \ldots$ a Markov chain?

In each case if your answer is yes give a short argument and specify the transition probabilities. If your answer is no give a specific violation of the Markov property.

2. A Markov chain on state space $\{1,2,3,4\}$ has transition matrix

$$\begin{pmatrix} 0 & 1/4 & 3/4 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

a) Draw the transition graph of the chain.

b) Find the following probabilities:

i) $\mathbb{P}(X_1 = 3 | X_0 = 2)$,

ii) $\mathbb{P}(X_2 = 3 | X_1 = 2)$,

iii) $\mathbb{P}(X_2 = 3 | X_1 = 2, X_0 = 1)$,

iv) $\mathbb{P}(X_2 = 3, X_1 = 2 | X_0 = 1)$.

3. Let $X_0, X_1, X_2, \ldots$ be the Markov chain on state space $\{1,2,3,4\}$ with the non-zero transition probabilities being $p_{11} = p_{44} = 1, p_{21} = p_{23} = p_{24} = 1/3, p_{31} = p_{32} = 1/2$.

a) Draw the transition graph.

b) Calculate the following:

i) $\mathbb{P}(X_n \neq 1,4 | X_0 = 2)$,

ii) $\mathbb{P}(X_n = 4 | X_0 = 2)$.

c) How do your answers in b) behave as n tends to infinity? What does this tell you about the process.

4. An insurance company offers a no claims discount system of 10% for each claim-free year up to a maximum of 50%. In year 1 a customer has no discount. At the end of each year in which the customer does not make a claim 10% is added to the discount, unless the discount is already 50% in which case there is no change. If the customer does make a claim then the discount in the following year is reduced to 0%. Suppose that the probability of a given customer making a claim in any year is p, and is independent of whether a claim is made in any other year.

i) Explain how to model the level of discount using a Markov chain.

ii) Draw the transition graph.

iii) What is the probabilty that the customer's discount in year 4 is 20%?

iv) What is the probability that the customer's discount in year 16 is 0%?

5. As in lectures consider a maze of n rooms with passages joining some of them. Define a random walk in the maze by starting in room 1 and at each step choosing a passage out of your current room uniformly at random with the condition that you are not allowed to return immediately to a room just visited. Let X_i be your room number after i steps
We showed in lectures that $X_0, X_1, X_2, \ldots$ is not a Markov chain. Give a Markov chain which does describe the process. [Hint: Modify S.]