SUMMARY SHEET: VECTORS
Engineering Maths II (MAE111), 2010

Vectors and Vector Algebra

Vector: A vector has **direction** and **magnitude**. Two vectors are **equal** if they have the same direction and the same magnitude. In a given situation (eg a physical situation with force vectors), two vectors are **equivalent** if they have the same effect.

Notation:

$$\vec{AB}, \boldsymbol{a}, \vec{a}, \underline{a}$$

For the magnitude of a vector, the following are used:

$$|\vec{AB}|, \quad |\boldsymbol{a}|, \quad |\vec{a}|$$

Remember, always distinguish a vector from a scalar!

Vector Addition: Vectors obey the parallelogram law of addition:

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$$

has the size and direction of the diagonal of the parallelogram formed from sides $\boldsymbol{a}$ and $\boldsymbol{b}$. Vector addition is *commutative*, and *associative*.

$$\begin{aligned} \boldsymbol{c} &= \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} \\ \boldsymbol{d} &= \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) \end{aligned}$$

Zero Vector, Negative Vectors, and Vector subtraction: The zero vector $\mathbf{0}$, is the vector which when added to another vector doesn't change it. Vector $-\boldsymbol{a}$ has the same magnitude as $\boldsymbol{a}$ but is in the opposite direction. Vector subraction is defined by addition of the negative vector:

$$\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$

Multiplication by scalar: The vector $m\boldsymbol{a}$, where m is a scalar, is the same direction as $\boldsymbol{a}$, but has a magnitude $|m||\boldsymbol{a}|$.

Vector Component Form

A **unit vector** is a vector with a magnitude of unity.

For $\widehat{\boldsymbol{\imath}}$ (called "i-hat") unit vector in x direction, $\widehat{\boldsymbol{\jmath}}$ unit vector in y and $\widehat{\boldsymbol{k}}$ unit vector in z direction, then for point P at (x, y, z)

$$\vec{OP} = x\widehat{\boldsymbol{\imath}} + y\widehat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}$$

and, similarly, any vector can be expressed as sum of components in x, y, and z directions.

Vectors are equal if all their components are also equal.

Addition and subtraction are done component by component. For example if $\boldsymbol{a} = 2\widehat{\boldsymbol{\imath}} - 3\widehat{\boldsymbol{\jmath}}$ and $\boldsymbol{b} = -4\widehat{\boldsymbol{\imath}} + \widehat{\boldsymbol{\jmath}}$, then $\boldsymbol{a} - \boldsymbol{b} = 6\widehat{\boldsymbol{\imath}} - 4\widehat{\boldsymbol{\jmath}}$.

Magnitude of a Vector, and Unit Vector: For a vector $\boldsymbol{v} = a\widehat{\boldsymbol{\imath}} + b\widehat{\boldsymbol{\jmath}} + c\widehat{\boldsymbol{k}}$ the magnitude is

$$v = |\boldsymbol{v}| = \sqrt{a^2 + b^2 + c^2}$$

The *unit vector* in the direction of $\boldsymbol{v}$ is

$$\widehat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{\sqrt{a^2 + b^2 + c^2}}\boldsymbol{v}$$

Direction Cosines: For vector $\boldsymbol{v} = a\widehat{\boldsymbol{\imath}} + b\widehat{\boldsymbol{\jmath}} + c\widehat{\boldsymbol{k}}$ The angle between the x axis and the vector $\boldsymbol{v}$ is α; the angle between y axis and $\boldsymbol{v}$ is β; the angle between z axis and $\boldsymbol{v}$ is γ. The direction cosines l, m and n are:

$$l = \cos\alpha = \frac{a}{|\boldsymbol{v}|}, \quad m = \cos\beta = \frac{b}{|\boldsymbol{v}|}, \quad n = \cos\gamma = \frac{c}{|\boldsymbol{v}|}.$$

Note:

$$l^2 + m^2 + n^2 = \frac{a^2 + b^2 + c^2}{|\boldsymbol{v}|^2} = 1$$

and $(l\widehat{\boldsymbol{\imath}} + m\widehat{\boldsymbol{\jmath}} + n\widehat{\boldsymbol{k}})$ is the unit vector in the direction of $\boldsymbol{v}$.

Equation of Line

Parametric vector form Vector $\boldsymbol{a}$ is a position vector of a point on a line, and $\boldsymbol{b}$ is parallel to the line. Then, *any* point of the line can be reached by moving to the point $\boldsymbol{a}$ on the line, and then moving by some appropriate amount parallel to the line. So, for an arbitrary point on the line with position vector $\boldsymbol{r}$:

$$\boldsymbol{r} = \boldsymbol{a} + s\boldsymbol{b}$$

where s is a parameter (i.e., some number which selects a particular point on the line). This is the parametric vector equation of a line.

Equation of Line: Cartesian form If an arbitrary point on the line has Cartesian coordinates (x, y, z), where the line has a vector equation $\boldsymbol{r} = \boldsymbol{a} + s\boldsymbol{b}$, and the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are:

$$\begin{aligned} \boldsymbol{a} &= a_1\widehat{\boldsymbol{\imath}} + a_2\widehat{\boldsymbol{\jmath}} + a_3\widehat{\boldsymbol{k}} \\ \boldsymbol{b} &= b_1\widehat{\boldsymbol{\imath}} + b_2\widehat{\boldsymbol{\jmath}} + b_3\widehat{\boldsymbol{k}} \end{aligned}$$

Then it follows, by eliminating the parameter, that

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

This is the *equation of the line in Cartesian coordinates.*

Special case: If $\boldsymbol{b}$ is a unit vector, then b_1, b_2, b_3 are the direction cosines of the line.

Special case: If the line passes through the origin, then $\boldsymbol{a} = \mathbf{0}$ and $a_1 = a_2 = a_3 = 0$.

Intersection of Two Lines: Two lines given by:

$$\boldsymbol{r}_1 = \boldsymbol{a} + s\boldsymbol{b}, \quad \boldsymbol{r}_2 = \boldsymbol{c} + t\boldsymbol{d}$$

will intersect if there are values of parameters s^* and t^* such that

$$\boldsymbol{r} = \boldsymbol{a} + s^*\,\boldsymbol{b} = \boldsymbol{c} + t^*\,\boldsymbol{d}$$

From the component forms for the vectors it follows that

$$a_1 + s^* \, b_1 = c_1 + t^* \, d_1$$
$$a_2 + s^* \, b_2 = c_2 + t^* \, d_2$$
$$a_3 + s^* \, b_3 = c_3 + t^* \, d_3$$

Two of these equations can be used to find values for s^* and t^*, but the lines will only truly intersect if the third equation is also satisfied for the same values of s^* and t^*. Thus it is important to check for intersection in all *three* components.

Multiplication of Vectors

There are **two** ways of multiplying vectors together: the scalar product (or dot product) produces a scalar (i.e. just a number), but the vector product produces a vector.

Vector Multiplication: Scalar Product

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \; |\boldsymbol{b}| \; \cos\theta$$

where θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Special case:

$$\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$$

Special case:

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0, \;\; \text{if} \;\; \boldsymbol{a} \perp \boldsymbol{b}$$

In component form, for $\boldsymbol{a} = a_1\widehat{\boldsymbol{\imath}} + a_2\widehat{\boldsymbol{\jmath}} + a_3\widehat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\widehat{\boldsymbol{\imath}} + b_2\widehat{\boldsymbol{\jmath}} + b_3\widehat{\boldsymbol{k}}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

So the rule is: multiply respective components and add.

Angle between vectors: From definitions:

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}| \; |\boldsymbol{b}|}$$

(Care has to be taken for negative $\cos\theta$.)

Vector Multiplication: Vector Product

$$\boldsymbol{a} \wedge \boldsymbol{b} = |\boldsymbol{a}| \; |\boldsymbol{b}| \; \sin\theta \; \widehat{\boldsymbol{e}}$$

Here θ is the angle from $\boldsymbol{a}$ to $\boldsymbol{b}$, and $\widehat{\boldsymbol{e}}$ is a unit vector perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$. The sense of direction of $\widehat{\boldsymbol{e}}$ is in a RH screw sense as when turning $\boldsymbol{a}$ towards $\boldsymbol{b}$.

Note that:

$$\boldsymbol{a} \wedge \boldsymbol{b} = -\boldsymbol{b} \wedge \boldsymbol{a}$$

Special case: Vector product is zero if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel:

$$\boldsymbol{a} \wedge \boldsymbol{b} = 0, \quad (\boldsymbol{a} \| \boldsymbol{b})$$

In component form:

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\widehat{\boldsymbol{\imath}} + (a_3 b_1 - a_1 b_3)\widehat{\boldsymbol{\jmath}} + (a_1 b_2 - a_2 b_1)\widehat{\boldsymbol{k}}$$

Alternatively, using determinant form:

$$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{vmatrix} \widehat{\boldsymbol{\imath}} & \widehat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Multiplication rules for Cartesian unit vectors

$$\widehat{\boldsymbol{\imath}} \cdot \widehat{\boldsymbol{\imath}} = \widehat{\boldsymbol{\jmath}} \cdot \widehat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{k}} \cdot \widehat{\boldsymbol{k}} = 1$$
$$\widehat{\boldsymbol{\imath}} \cdot \widehat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{\imath}} \cdot \widehat{\boldsymbol{k}} = \widehat{\boldsymbol{\jmath}} \cdot \widehat{\boldsymbol{k}} = 0$$

$$\widehat{\boldsymbol{\imath}} \wedge \widehat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{k}}$$
$$\widehat{\boldsymbol{\jmath}} \wedge \widehat{\boldsymbol{k}} = \widehat{\boldsymbol{\imath}}$$
$$\widehat{\boldsymbol{k}} \wedge \widehat{\boldsymbol{\imath}} = \widehat{\boldsymbol{\jmath}}$$
$$\widehat{\boldsymbol{\imath}} \wedge \widehat{\boldsymbol{\imath}} = \widehat{\boldsymbol{\jmath}} \wedge \widehat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{k}} \wedge \widehat{\boldsymbol{k}} = 0$$

Applications of Vectors

Area of triangle: where two sides of triangle are represented by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$A = \frac{1}{2}|\boldsymbol{a} \wedge \boldsymbol{b}|$$

Work done by a force The work W (a scalar) done by a force $\boldsymbol{F}$ (a vector) which causes a displacement $\boldsymbol{s}$ (a vector) is given by

$$W = \boldsymbol{F} \cdot \boldsymbol{s}$$

Moment of force If a force $\boldsymbol{F}$ acts at some point given by position vector $\boldsymbol{a}$, then the moment of the force about the origin is defined as

$$\boldsymbol{G} = \boldsymbol{a} \wedge \boldsymbol{F}$$

Centre of Mass of collection of point masses For a set of n point masses with masses $m_1, m_2, m_3, \ldots m_n$ which lie in the x-y plane with positions (x_i, y_i) so that their position vectors are $\boldsymbol{r}_i = x_i\widehat{\boldsymbol{\imath}} + y_i\widehat{\boldsymbol{\jmath}}$, then their centre of mass position is given by:

$$\boldsymbol{r}_{\text{CM}} = \frac{1}{M}\sum_{i=1}^{n} m_i \boldsymbol{r}_i$$

where M is their total mass: $M = \sum_i m_i$.

Text Book References

	Engineering Mathematics Croft Davison Hargreaves	Mathematics for Engineers Croft & Davison	Engineering Mathematics Stroud (4th, 5th,6th Edition)
Vectors	Ch 7	Ch 9	Programme 6