MAE111 Engineering Mathematics II
(2003/2004 Sem. 1)

SECTION 1: VECTORS

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If you find a mistake, then please tell the author!

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Section 1

Vectors

1.1 Quantities with Magnitude and Direction

⇓ .... *Hour 1* ⇓

We are used to using numbers to specify quantities in everyday situations. We might ask for 3 kg of apples, or see that the speed limit is 40 mph. In engineering, similarly, the value of various quantities, such as mass, temperature, voltage, current, etc., is specified with a number.

On the other hand, there are some situations where the use of a single number seems inadequate. Think about the problem of getting directions to reach a certain place. If we are told just the distance, then it is not sufficient, because we also need the direction. “Go three miles.” “Yes, but in what direction?” “Go South.” “OK, but how far?”

In this case the distance and the direction are completely linked. One without the other simply doesn’t make sense. To describe a movement, or displacement, from one place to another needs a distance (i.e., a number) and a direction. There are many other such examples from engineering. A force of a given magnitude acts in a certain direction. The effect of the wind is measured by its direction and speed.

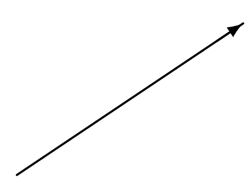
We want a mathematical way to deal with these ideas, namely something that deals with both a number giving size (or magnitude) and a direction.

Vectors are the mathematical objects which can be used for this problem. They can be regarded as an extension of the rules of arithmetic for numbers to deal with the geometry of directions.

A vector can be represented as a “directed line segment.”

In other words . . . an arrow!

The arrow gives you **direction** and its length represents the vector’s **magnitude**.



A vector has a *magnitude*, like a number, but it also has a *direction*. Vectors can be used to represent different physical quantities, such a velocity, magnetic field and so on, but its mathematical properties depend only on its two properties of magnitude and direction. From this basic assumption a self-consistent set of rules, just like the rules of arithmetic, can be developed for operating on, and combining vectors.

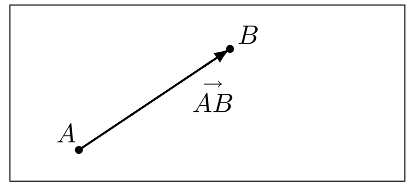
The set of rules for combining and operating on vectors is called **vector algebra.**

Properties which require both magnitude and direction in order to specify them are called **vector quantities**, or just vectors. Properties that require just a single number to specify magnitude are known as **scalar quantities** or just scalars.

Scalar Quantities	Vector Quantities
Temperature	Position
Voltage	Force
Mass	Magnetic field
Speed	Velocity
Number of people	

1.2 Vector Algebra

1.2.1 Notation



The simplest representation of direction and magnitude is the line segment between two points, say A and B. The corresponding vector can be written

$$\vec{AB}$$

Note that the sense of direction is from A to B.

Symbolically a vector may be represented as the vector $\boldsymbol{a}$ which is in **bold face** in printed text, or $\vec{a}$ or $\underline{a}$ in written text.

The magnitude of a vector (which is a scalar quantity) is shown as:

$$|\vec{AB}|, \quad |\boldsymbol{a}|, \quad |\vec{a}|$$

and so on.

1.2.2 Equality and Equivalence

Key Point

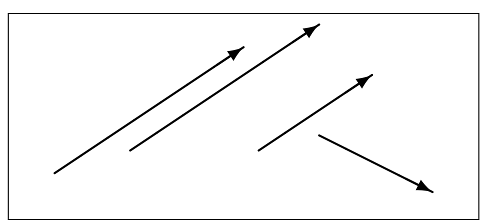
Equal vectors have the same direction and the same magnitude

Mathematically, a vector has only direction and magnitude. It follows that:

Two vectors are equal if, and only if, they have equal direction and equal magnitude.

Two equal vectors, and two unequal vectors.

Note that equal vectors are, obviously, parallel to each other.



However, there are obviously engineering or physical situations where this strict definition of equality must be extended. For example, if looking at the equality of forces, then the dimensions (i.e., units) of the vector quantities might be important. But there is a further important issue.

Consider two forces, which can be represented by vectors of equal magnitude and direction. These two forces act at different places on a beam, and will produce different effects (they produce different moments). Although the vectors are *equal* they are not *equivalent*.

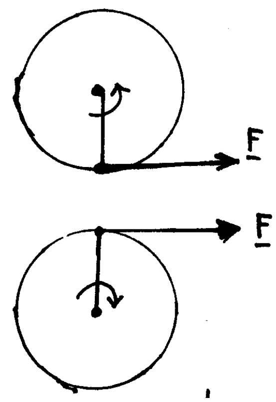
- *Equality of vectors is determined by direction and magnitude.*
- *Equivalence of effect depends on the particular situation.*

It is important to distinguish between times when a vector can be treated as only direction and magnitude, and other times when the situation imposes further constraints. Examples of the latter case may be: a force acting at a point, a vector used to represent a position, etc.

The figure show equal vectors, but the forces they represent have different effects.

In this case the force vectors are **non-equivalent**.

Notice how they are called "force vectors" to show that they represent forces which (usually) act at a given point.



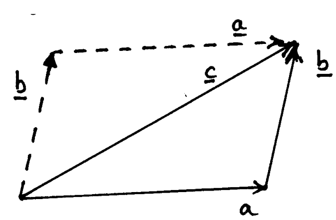
1.2.3 Vector Addition

Vectors obey what is called the **parallelogram law of addition.**

The vector

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$$

has the size and direction of the diagonal of the parallelogram formed from sides $\boldsymbol{a}$ and $\boldsymbol{b}$.



In terms of displacements, moving by the displacements represented by $\boldsymbol{a}$ and then $\boldsymbol{b}$ has the same effect (the same end point) as moving by the displacement represented by $\boldsymbol{c}$. Clearly, the order of the $\boldsymbol{a}$ and $\boldsymbol{b}$ displacments can be reversed without changing the result.

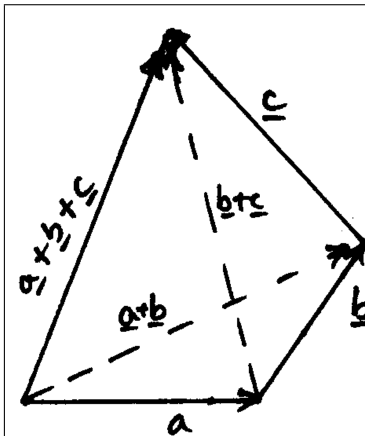
$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$

So that vector addition is *commutative*.

We can also consider the addition of three vectors, and from applying the parallelogram rule, we see that:

$$\begin{aligned} \boldsymbol{d} &= \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} \\ &= (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} \\ &= \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) \end{aligned}$$

So that vector addition is *associative*.



1.2.4 Null Vector

Key Point

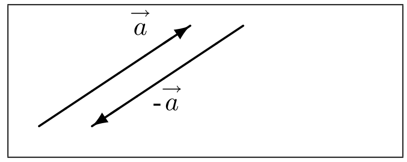
The null vector is not a scalar.

By analogy to the ordinary numbers and zero, we can introduce the **null vector** $\mathbf{0}$ which has zero magnitude (and arbitrary direction). Its definition is that it is the vector which, added to any vector, produces no change.

$$\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$$

Note: the null vector is **not** a scalar!

1.2.5 Negative Vectors and Vector Subtraction



The negative of a vector, written $(-\boldsymbol{a})$, is a vector of equal magnitude, by directly opposite direction. Thus the effect of adding a vector and its negative is the null vector.

$$\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$$

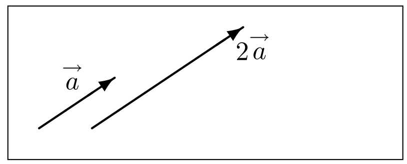
For line segments we can write:

$$-(\vec{AB}) = \vec{BA}$$

Vector subtraction follows as the addition of the negative of the vector:

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$

1.2.6 Multiplication by Scalar



The quantity

$$m\boldsymbol{A}$$

where m is a scalar, is a vector with the same direction as $\boldsymbol{A}$, but with a magnitude

$$|m|\ |\boldsymbol{A}|$$

If m is positive then $m\boldsymbol{A}$ is in the same direction as $\boldsymbol{A}$, but in the opposite direction if m is negative. Thus if $m = -1$, then we get consistency with negative vectors:

$$(-1)\boldsymbol{A} = -\boldsymbol{A}$$

Key Point

Multiplying by a scalar changes magnitude, but not direction.

Multiplication by a scalar follows these (rather obvious) rules:

$$m(\boldsymbol{A} + \boldsymbol{B}) = m\boldsymbol{A} + m\boldsymbol{B}$$

$$(m + n)\boldsymbol{A} = m\boldsymbol{A} + n\boldsymbol{A}$$

$$m(n\boldsymbol{A}) = n(m\boldsymbol{A}) = mn\boldsymbol{A}$$

Also, a vector multiplied by zero, produces the null vector:

$$(0)\ \boldsymbol{A} = \mathbf{0}$$

1.3 Vectors and Geometry

Vectors are represented on paper as line segments drawn to scale. It follows that we can express a lot about *geometry*, or the relationships of points and lines in space, in terms of vectors.

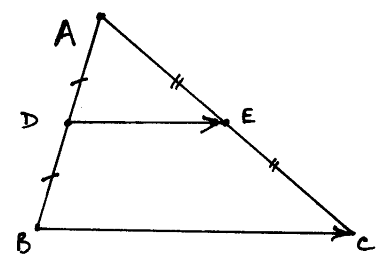
Example

A triangle ABC has points D and E which are the midpoints of AB and AC.

Prove that DE is parallel to BC and half the length.

Applying the vector law of addition, and then using some vector algebra:

$$\vec{DE} = \vec{DA} + \vec{AE} = \frac{1}{2}\vec{BA} + \frac{1}{2}\vec{AC} = \frac{1}{2}(\vec{BA} + \vec{AC}) = \frac{1}{2}\vec{BC}$$



So, $\vec{DE}$ is a scalar multiple of $\vec{BC}$, so the DE must be parallel to BC, since they have the same direction. And,

$$|\vec{DE}| = \frac{1}{2}\,|\vec{BC}|$$

so that DE is half the length of BC, since the magnitude of $\vec{DE}$ is half that of $\vec{BC}$.

Example

Consider two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ which are perpendicular. Their sum is the vector $\boldsymbol{c}$:

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$$

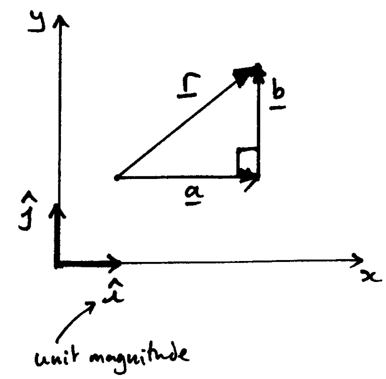
Because $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular we can use everything we know about right angle triangles:

$$\begin{aligned} |\boldsymbol{c}| &= |\boldsymbol{a}+\boldsymbol{b}| = \sqrt{|\boldsymbol{a}| + |\boldsymbol{b}|} \\ \cos\theta &= \frac{|\boldsymbol{a}|}{|\boldsymbol{c}|} \\ \textit{etc.}\ \ldots \end{aligned}$$

(where θ is the angle between $\boldsymbol{c}$ and $\boldsymbol{a}$).

⇓ .... Hour 2 ⇓

1.4 Vector Component Form



For a vector $\boldsymbol{r}$ there are always two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that

$$\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b}$$

and $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other.

Now, if there is a fixed coordinate frame whose x axis is parallel to $\boldsymbol{a}$, and whose y axis is parallel to $\boldsymbol{b}$, then we can rewrite $\boldsymbol{a}$ and $\boldsymbol{b}$ in terms of the unit vectors which define the two axes. So, a unit vector in the x direction is defined as $\widehat{\boldsymbol{\imath}}$ (sometimes written as $\widehat{\boldsymbol{x}}$), and similarly $\widehat{\boldsymbol{\jmath}}$ is defined as the unit vector in the y direction.

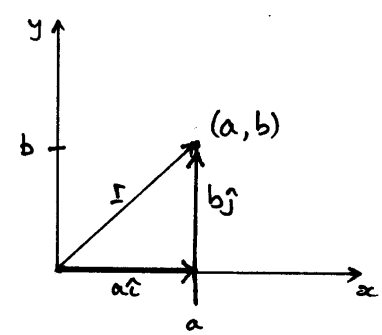
We define the following unit vectors:
$\widehat{\boldsymbol{\imath}}$: vector in x direction with unit magnitude
$\widehat{\boldsymbol{\jmath}}$: vector in y direction with unit magnitude

Then, if a and b are the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$, we can write

$$\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b} = a\widehat{\boldsymbol{\imath}} + b\widehat{\boldsymbol{\jmath}}.$$

The scalar quantities a and b are called the cartesian *components* of the vector $\boldsymbol{r}$ in the x and y directions, respectively. Cartesian simply means that the coordinate system axes are perpendicular to each other and do not change.

1.4.1 Position vectors



Vectors can also be represented as a displacement from one point to another. If the initial point is the origin of a coordinate system, then we can use vectors to “label” points in space. Such a vector is called a **position vector**.

Note that position vectors are an example of a specialized form of vector: they have direction and magnitude, but they also have an extra meaning (namely, the displacement from an origin to a point in space).

There is an obvious relationship between the coordinates of a point and the corresponding position vector.

In two dimensions (2D) the point with coordinates $x = a$ and $y = b$, or (a, b) has a position vector $\boldsymbol{r}$.

From the vector law for addition we find the relationship between a point’s coordinates and its position vector:

$$\boldsymbol{r} = a\widehat{\boldsymbol{\imath}} + b\widehat{\boldsymbol{\jmath}}$$

From this we can see that there is a correspondence between the *coordinates* of a position vector and the *components* of a vector. This can be confusing, and one has to take care to distinguish position vectors, which indicate displacement from the origin, from other vectors, such as ones giving merely a relative displacement between two different points.

1.4.2 Vector equality with components

Key Point

Equal vectors are equal component by component.

From the definition of components it obviously follows that two vectors are *equal* if their x components are equal and also their y components are equal.

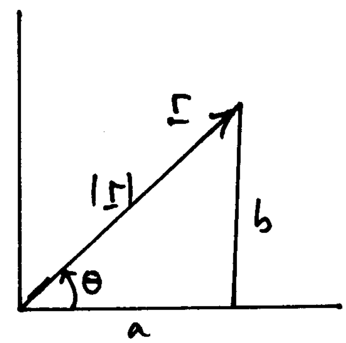
Consider two vectors $\boldsymbol{s}$ and $\boldsymbol{t}$ with components

$$\begin{aligned} \boldsymbol{s} &= p\widehat{\boldsymbol{\imath}} + q\widehat{\boldsymbol{\jmath}} \\ \boldsymbol{t} &= u\widehat{\boldsymbol{\imath}} + v\widehat{\boldsymbol{\jmath}} \end{aligned}$$

In order for $\boldsymbol{s}$ and $\boldsymbol{t}$ to be equal, the following must be true:

$$\begin{aligned} p &= u \\ q &= v \end{aligned}$$

1.4.3 Magnitude and direction using components



Plotting a position vector $\boldsymbol{r}$ for a coordinate (a, b)

we can see that the lengths r (i.e., $|\boldsymbol{r}|$ the magnitude of the vector $\boldsymbol{r}$), a and b form a right angled triangle, so using Pythagoras:

$$r^2 = a^2 + b^2,$$

or

$$r = |\boldsymbol{r}| = \sqrt{a^2 + b^2}.$$

This gives us a way to calculate the magnitude of a vector in terms of its components.

Also, if the vector $\boldsymbol{r}$ makes an angle θ to the x axis, then

$$\cos\theta = \frac{a}{r} = \frac{a}{|\boldsymbol{r}|} = \frac{a}{\sqrt{a^2 + b^2}}.$$

so that the direction of the vector, in terms of $\cos\theta$, can be calculated from the components of $\boldsymbol{r}$.

Example

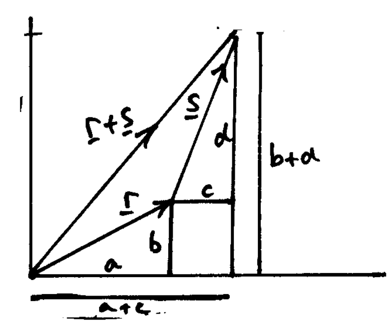
Given a vector $\boldsymbol{r} = 3\widehat{\boldsymbol{\imath}} + 4\widehat{\boldsymbol{\jmath}}$, its magnitude is found from

$$r = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5.$$

The angle to the x axis can be found by

$$\theta = \cos^{-1}\left(\frac{3}{5}\right) = 53.13^\circ = 0.927\text{radian}.$$

1.4.4 Vector addition and subtraction using components



Suppose we have two vectors $\boldsymbol{r} = a\widehat{\boldsymbol{\imath}} + b\widehat{\boldsymbol{\jmath}}$ and $\boldsymbol{s} = c\widehat{\boldsymbol{\imath}} + d\widehat{\boldsymbol{\jmath}}$, and we wish to find the vector which is the sum of these two vectors: $\boldsymbol{p} = \boldsymbol{r} + \boldsymbol{s}$. By constructing the triangle of vector addition, we can see that the x component of $\boldsymbol{p}$ is the sum of the x components of $\boldsymbol{r}$ and $\boldsymbol{s}$, and similarly for the y component.

$$\boldsymbol{p} = \boldsymbol{r} + \boldsymbol{s} = a\widehat{\boldsymbol{\imath}} + b\widehat{\boldsymbol{\jmath}} + c\widehat{\boldsymbol{\imath}} + d\widehat{\boldsymbol{\jmath}} = (a + c)\widehat{\boldsymbol{\imath}} + (b + d)\widehat{\boldsymbol{\jmath}}.$$

Thus the rule for addition (or subtraction) is to proceed component by component.

Key Point

For vector addition, add component by component.

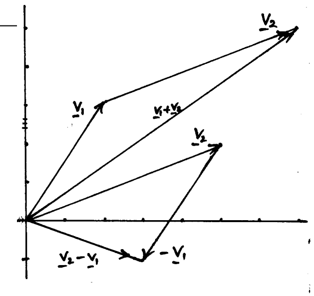
:ample

Consider (in 2D)

$$\boldsymbol{v}_1 = 2\hat{\imath} + 3\hat{\jmath}, \quad \boldsymbol{v}_2 = 5\hat{\imath} + 2\hat{\jmath}$$

Then

$$\boldsymbol{v}_1 + \boldsymbol{v}_2 = (2+5)\hat{\imath} + (3+2)\hat{\jmath} = 7\hat{\imath} + 5\hat{\jmath}$$
$$\boldsymbol{v}_2 - \boldsymbol{v}_1 = (5-2)\hat{\imath} + (2-3)\hat{\jmath} = 3\hat{\imath} - \hat{\jmath}$$



1.5 Using Vectors: Two Examples

1.5.1 Vector Addition of Velocities

A swimmer can swim in still water at 3 m/s.

They try to cross a river which is flowing at 4 m/s, and they swim perpendicular to the direction of flow.

Their velocity $\boldsymbol{V}$, as seen from the river bank is the vector sum of the river's flow velocity $\boldsymbol{V}_r$ and the swimmer's still water velocity $\boldsymbol{V}_s$ (which has magnitude 3 m/s and is perpendicular to $\boldsymbol{V}_r$).

$$\boldsymbol{V} = \boldsymbol{V}_r + \boldsymbol{V}_s$$

Since $\boldsymbol{V}_r$ and $\boldsymbol{V}_s$ are perpendicular, we can arrange suitable coordinate unit vectors $\hat{\imath}$ and $\hat{\jmath}$ so that

$$\boldsymbol{V} = 4\hat{\imath} + 3\hat{\jmath}$$

Straight away the swimmer's speed can be calculated:

$$|\boldsymbol{V}| = \sqrt{4^2 + 3^2} = 5.$$

Once this problem has been set up, other things can be done. For example if there a wind velocity measured on the bank, what is the effective wind velocity felt by the swimmer?

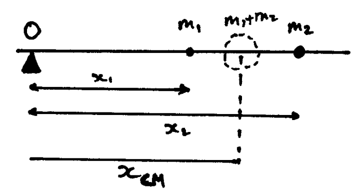
⇓ . . . . *Hour 3* ⇓

1.5.2 Centre of Mass of Collection of Point Masses

We will consider a set of so-called point masses, arranged in a plane so that we can use two dimensional vectors. A point mass is a convenient fiction, so that a particular mass can be thought of as existing only at a single point in space.

The **centre of mass** is the point about which the turning moment is zero – in other words, the effects of the masses about this point balance out.

Alternatively . . . The total turning moment about some point is the same as if all the mass is concentrated at the centre of mass.



The turning moment of a mass about a point is the product of the mass and the distance of the mass from the point in question.

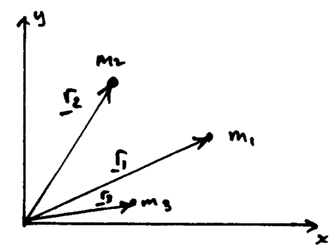
First think about a one dimensional arrangement (i.e., the masses are arranged in a line) with two masses m_1 and m_2. The position of m_1 is given by its distance x_1 from the origin O, and similarly m_2 is a distance x_2 from O.

The centre of mass has the following property: The combined moment of m_1 and m_2 about O is the same as the moment of the combined mass $(m_1 + m_2)$ if it were situated at the position of the centre of mass, x_{CM}. As far as the moment is concerned we can replace the two masses by a single mass $(m_1 + m_2)$ at the centre of mass.

$$m_1 x_1 + m_2 x_2 = (m_1 + m_2) x_{\text{CM}}$$

So,

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$



This can easily be generalized to the case where there are many, say n, such point masses $m_1, m_2, m_3, \ldots m_n$ positioned at $x_1, x_2, x_3, \ldots x_n$,

$$x_{\text{CM}} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i$$

where the total mass is $M = \sum_i m_i$.

If the point masses are arranged in the x-y plane, with y coordinates $y_1, y_2, y_3, \ldots y_n$, then the above formula is correct from taking the moment about the y axis and hence finding the x coordinate of the centre of mass. By taking the moment about the x axis we can similarly find the y coordinate:

$$y_{\text{CM}} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i$$

Now, each mass m_i is at position (x_i, y_i) so the corresponding *position vector* is $\boldsymbol{r}_i = x_i \hat{\boldsymbol{\imath}} + y_i \hat{\boldsymbol{\jmath}}$. Also we can write the position vector of the centre of mass as $\boldsymbol{r}_{\text{CM}} = x_{\text{CM}} \hat{\boldsymbol{\imath}} + y_{\text{CM}} \hat{\boldsymbol{\jmath}}$. Thus,

$$\begin{aligned} M \boldsymbol{r}_{\text{CM}} &= \left(\sum_i m_i x_i \right) \hat{\boldsymbol{\imath}} + \left(\sum_i m_i y_i \right) \hat{\boldsymbol{\jmath}} \\ &= \sum_i (m_i x_i \hat{\boldsymbol{\imath}} + m_i y_i \hat{\boldsymbol{\jmath}}) \\ &= \sum_i m_i \boldsymbol{r}_i \end{aligned}$$

Finally:

$$\boldsymbol{r}_{\text{CM}} = \frac{1}{M} \sum_i m_i \boldsymbol{r}_i$$

The centre of mass position vector is the mass weighted average of the position vectors of all the point masses.

Note that this vector equation is independent of any coordinate frame, but to calculate $\boldsymbol{r}_{\text{CM}}$ we need to use a particualr coordinate frame so that we can use components.

Example

Masses of 3g, 5g, and 1g are situated at position vectors $\boldsymbol{a} = 2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$, $\boldsymbol{b} = -3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$, and $\boldsymbol{c} = -8\hat{\boldsymbol{\jmath}}$, respectively.

The total mass:

$$M = 3 + 5 + 1 = 9$$

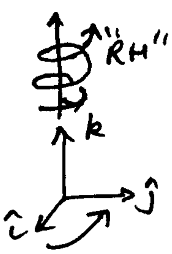
So, applying the above result:

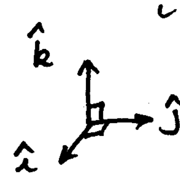
$$\begin{aligned} 9 \boldsymbol{r}_{\text{CM}} &= (3 \cdot 2 - 5 \cdot 3) \hat{\boldsymbol{\imath}} + (3 \cdot 4 + 5 \cdot 1 - 1 \cdot 8) \hat{\boldsymbol{\jmath}} \\ &= (6 - 15) \hat{\boldsymbol{\imath}} + (12 + 5 - 9) \hat{\boldsymbol{\jmath}} \\ &= -9 \hat{\boldsymbol{\imath}} + 9 \hat{\boldsymbol{\jmath}} \end{aligned}$$

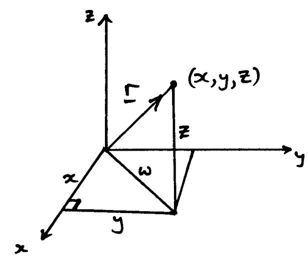
And finally:

$$\boldsymbol{r}_{\text{CM}} = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$$

1.6 Three Dimensional Vector Component Form







Everything that has been given for the case of a two dimensional x-y coordinate system can be extended to a three dimensional (3D) Cartesian coordinate system: choosing a fixed set of orthogonal (right-handed) axes, then defining the following unit vectors:
$\widehat{\boldsymbol{\imath}}$: vector in x direction with unit magnitude
$\widehat{\boldsymbol{\jmath}}$: vector in y direction with unit magnitude
$\widehat{\boldsymbol{k}}$: vector in z direction with unit magnitude

Then the position vector corresponding to the point P at (x, y, z) can be written as

$$\overrightarrow{OP} = x\widehat{\boldsymbol{\imath}} + y\widehat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}$$

This is known as *vector component form.*

1.6.1 Equality and Addition

If two vectors are equal:

$$\begin{aligned} \boldsymbol{p} &= \boldsymbol{r} \\ p_x\widehat{\boldsymbol{\imath}} + p_y\widehat{\boldsymbol{\jmath}} + p_z\widehat{\boldsymbol{k}} &= r_x\widehat{\boldsymbol{\imath}} + r_y\widehat{\boldsymbol{\jmath}} + r_z\widehat{\boldsymbol{k}} \end{aligned}$$

then the components are equal:

$$p_x = r_x, \quad p_y = r_y, \quad p_z = r_z$$

Suppose there is a vector

$$\overrightarrow{OQ} = a\widehat{\boldsymbol{\imath}} + b\widehat{\boldsymbol{\jmath}} + c\widehat{\boldsymbol{k}}$$

Then the effect of adding the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ must be the effect of doing the first displacement, followed by the second. In other words:

$$\overrightarrow{OP} + \overrightarrow{OQ} = (x+a)\widehat{\boldsymbol{\imath}} + (y+b)\widehat{\boldsymbol{\jmath}} + (z+c)\widehat{\boldsymbol{k}}$$

So, to add vectors in component form: add the corresponding components.

1.6.2 Magnitude of a Vector

From the diagram, we have $w^2 = x^2 + y^2$ (by Pythagoras). Also, if the vector has magnitude r

$$r^2 = |\boldsymbol{r}|^2 = w^2 + z^2 = x^2 + y^2 + z^2$$

So,

$$r = |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$$

So, for a general vector

$$\boldsymbol{v} = a\widehat{\boldsymbol{\imath}} + b\widehat{\boldsymbol{\jmath}} + c\widehat{\boldsymbol{k}}$$

has magnitude

$$v = |\boldsymbol{v}| = \sqrt{a^2 + b^2 + c^2}$$

It follows that the *unit vector* in the direction of $\boldsymbol{v}$ is

$$\widehat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{\sqrt{a^2 + b^2 + c^2}}\boldsymbol{v}$$

1.7 Vector Multiplication

⇓ . . . . *Hour 4* ⇓

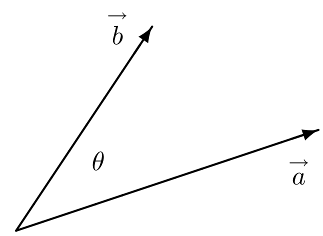
We have seen that a logical meaning can be given to multiplication of a vector by a scalar. But what is the meaning of multiplying two vectors together? How should we interpret the multiplication of a direction by another direction? It turns out that one can define two completely different kinds of vector multiplication:

- Scalar product (or "dot" product): the scalar product of two vectors is a scalar (ie ordinary number), so: vector · vector = scalar
- Vector product (or "cross" product): the vector product of two vectors is another vector, so: vector ∧ vector = vector

1.7.1 Scalar Product, or "Dot" Product

The definition is of the scalar product of two vectors is

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}|\ \cos\theta$$



where θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

The definition is

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}|\ \cos\theta$$

where θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

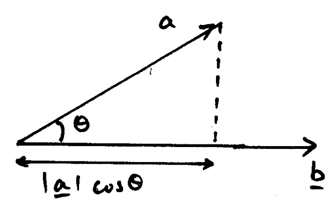
From the definition we have some obvious consequences:

$$\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$$

so the order of multiplication of a scalar product does not affect the result.

The usual rules of algebra apply for expansions:

$$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$$



Geometrically, $|\boldsymbol{a}|\cos\theta$ is the component of the vector $\boldsymbol{a}$ in the direction of the vector $\boldsymbol{b}$, which is also known as the projection of the vector $\boldsymbol{a}$ on to the vector $\boldsymbol{b}$.

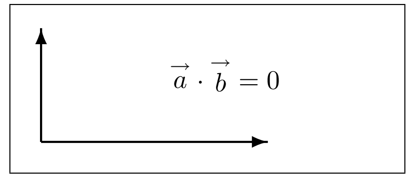
$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{b}|(\text{projection of } \boldsymbol{a} \text{ on to } \boldsymbol{b}) = |\boldsymbol{a}|(\text{projection of } \boldsymbol{b} \text{ on to } \boldsymbol{c})$$

Importantly, the scalar product of a vector with itself is equal to the square of the magnitude of the vector (since the angle between the vectors is zero):

$$\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$$

If the two vectors are perpendicular, so that the angle between them is 90°, then the scalar product is zero:

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0, \quad \text{if} \;\; \boldsymbol{a} \perp \boldsymbol{b}$$



Scalar Product in Component Form

Suppose that

$$\boldsymbol{a} = a_1\widehat{\boldsymbol{\imath}} + a_2\widehat{\boldsymbol{\jmath}} + a_3\widehat{\boldsymbol{k}}$$
$$\boldsymbol{b} = b_1\widehat{\boldsymbol{\imath}} + b_2\widehat{\boldsymbol{\jmath}} + b_3\widehat{\boldsymbol{k}}$$

Then

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_1\widehat{\boldsymbol{\imath}} + a_2\widehat{\boldsymbol{\jmath}} + a_3\widehat{\boldsymbol{k}}) \cdot (b_1\widehat{\boldsymbol{\imath}} + b_2\widehat{\boldsymbol{\jmath}} + b_3\widehat{\boldsymbol{k}})$$

Now, $\widehat{\boldsymbol{\imath}}$ is a unit vector, so

$$\widehat{\boldsymbol{\imath}} \cdot \widehat{\boldsymbol{\imath}} = 1$$

And, by definition, $\widehat{\boldsymbol{\imath}}$ is perpendicular to $\widehat{\boldsymbol{\jmath}}$ and $\widehat{\boldsymbol{k}}$ (since the three unit vectors are mutually orthogonal), so

$$\widehat{\boldsymbol{\imath}} \cdot \widehat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{\imath}} \cdot \widehat{\boldsymbol{k}} = 0$$

It follows that

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

So the rule is: multiply respective components and add.

Key Point

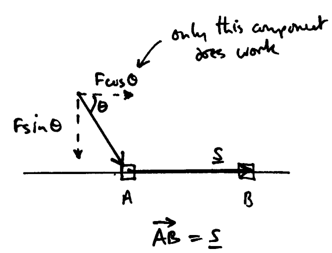
For dot product, multiply component by component and then add up.

Work Done by a Force

Consider a vector force $\boldsymbol{F}$ applied to an object which moves from point A to B, which is represented by a displacement vector $\boldsymbol{s}$.

The work done, W, by a force is equal to the magnitude of the force multiplied by the distance moved in the direction of that force. The component of the force perpendicular to the displacement thus does no work. The component of $\boldsymbol{F}$ parallel to the displacement $\boldsymbol{s}$ is $|\boldsymbol{F}|\cos\theta$, where θ is the angle between $\boldsymbol{F}$ and $\boldsymbol{s}$

$$W = |\boldsymbol{F}| \;\cos\theta \;|\boldsymbol{s}| = \boldsymbol{F} \cdot \boldsymbol{s}$$



(This equation strictly only applies for constant $\boldsymbol{F}$.)

Example

A force $\boldsymbol{F}$

$$\boldsymbol{F} = 3\widehat{\boldsymbol{\imath}} + 2\widehat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}$$

is applied to an object which moves a vector distance $\boldsymbol{s}$:

$$\boldsymbol{s} = 2\widehat{\boldsymbol{\imath}} + 2\widehat{\boldsymbol{\jmath}} + \widehat{\boldsymbol{k}}$$

What is the work done by the force?

Work done, W:

$$W = \boldsymbol{F} \cdot \boldsymbol{s} = (3)(2) + (2)(2) + (-1)(1) = 6 + 4 - 1 = 9 \quad \text{(Joule)}$$

Note that $\boldsymbol{F}$ should have units Newton (N), and $\boldsymbol{s}$ should have units in metres (m).

Angle between vectors

The component form of the scalar product can be used to find the angle between two vectors, using

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\ |\boldsymbol{b}|}$$

Note that special care has to be taken for negative $\cos\theta$ to find the correct value for θ.

Key Point

Angle between vectors:

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\ |\boldsymbol{b}|}$$

Example

Find the angle between the vectors

$$\boldsymbol{a} = 5\widehat{\boldsymbol{\imath}} + 2\widehat{\boldsymbol{\jmath}} - 7\widehat{\boldsymbol{k}}$$

$$\boldsymbol{b} = 2\widehat{\boldsymbol{\imath}} - \widehat{\boldsymbol{\jmath}} + 3\widehat{\boldsymbol{k}}$$

Finding the magnitudes:

$$|\boldsymbol{a}| = (5^2 + 2^2 + (-7)^2)^{1/2} = \sqrt{78}$$

$$|\boldsymbol{b}| = (4 + 1 + 9)^{1/2} = \sqrt{14}$$

The dot product is

$$\boldsymbol{a}\cdot\boldsymbol{b} = (5)(2) + (2)(-1)(-7)(3) = 10 - 2 - 21 = -13$$

So,

$$\cos\theta = \frac{-13}{\sqrt{78}\,\sqrt{14}} = -0.3934$$

Therefore,

$$\theta = 113.17^\circ$$

The Cosine Rule

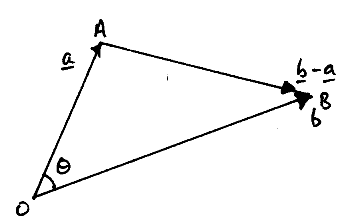
Another link between geometry and vectors can be made using the scalar product.

Consider the triangle OAB where θ is the angle between OA and OB. We can write:

$$\boldsymbol{a} = \overrightarrow{OA}, \quad \boldsymbol{b} = \overrightarrow{OB}, \quad \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$$

It then follows that

$$\begin{aligned} |\overrightarrow{AB}|^2 &= (\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a}) \\ &= \boldsymbol{a}\cdot\boldsymbol{a} + \boldsymbol{b}\cdot\boldsymbol{b} - 2\boldsymbol{a}\cdot\boldsymbol{b} \\ &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}|\ |\boldsymbol{b}|\ \cos\theta \end{aligned}$$

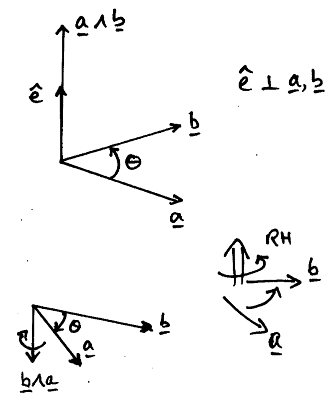


This is none other than the *Cosine Rule* for the triangle OAB:

$$AB^2 = OA^2 + OB^2 - 2(OA)\ (OB)\ \cos\theta$$

⇓ . . . . *Hour 5* ⇓

1.7.2 Vector Product, or Cross Product



The vector product of two vectors produces a third vector which is perpendicular to both. The magnitude is equal to the product of the magnitudes multiplied by the sine of the angle from the first vector to the second.

The definition is thus:

$$\boldsymbol{a} \wedge \boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}|\ \sin\theta\, \widehat{\boldsymbol{e}}$$

Here θ is the angle from $\boldsymbol{a}$ to $\boldsymbol{b}$, and $\widehat{\boldsymbol{e}}$ is a unit vector perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$. The sense of direction of $\widehat{\boldsymbol{e}}$ is in a RH screw sense as when turning $\boldsymbol{a}$ towards $\boldsymbol{b}$.

The vector product is pronounced “a cross b,” and may be written $\boldsymbol{a} \times \boldsymbol{b}$. The cross product is a completely different kind of multiplication, and so must always have the correct symbol.

From the definition of the sense of $\widehat{\boldsymbol{e}}$ it is apparent that

$$\boldsymbol{a} \wedge \boldsymbol{b} = -\boldsymbol{b} \wedge \boldsymbol{a}$$

(ie changing the order in a vector product changes the direction of the result). This is the most important difference with ordinary multiplication of numbers: the vector product is non-commutative.

Other rules of algebra are more usual:

$$\boldsymbol{a} \wedge (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \wedge \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{c}$$

$$k(\boldsymbol{a} \wedge \boldsymbol{b}) = (k\boldsymbol{a}) \wedge \boldsymbol{b} = \boldsymbol{b}) \wedge (k\boldsymbol{b})$$

where k is a scalar.

If two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel then, one can find the unit vector normal to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\widehat{\boldsymbol{e}} = \frac{\boldsymbol{a} \wedge \boldsymbol{b}}{|\boldsymbol{a} \wedge \boldsymbol{b}|}$$

Special Cases

If $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$, then the angle $\theta = 0$, so that $\sin\theta = 0$, so that

$$\boldsymbol{a} \wedge \boldsymbol{b} = 0, \quad (\boldsymbol{a} \| \boldsymbol{b})$$

On the other hand, if $\boldsymbol{a} \wedge \boldsymbol{b} = 0$ this implies one or more possibilities:

- Either $\boldsymbol{a} = \boldsymbol{0}$ and/or $\boldsymbol{b} = \boldsymbol{0}$
- Or $\boldsymbol{a} \| \boldsymbol{b}$ or $\boldsymbol{a} \| (-\boldsymbol{b})$

If $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ then the angle $\theta = 90°$, so

$$\boldsymbol{a} \wedge \boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}|, \quad (\boldsymbol{a} \perp \boldsymbol{b})$$

Vector Product in Component Form

Applying the vector product to the Cartesian unit vectors we find:

$$\begin{array}{ll}\widehat{\boldsymbol{\imath}} \wedge \widehat{\boldsymbol{\imath}} = \widehat{\boldsymbol{\jmath}} \wedge \widehat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{k}} \wedge \widehat{\boldsymbol{k}} = 0 & \\ \widehat{\boldsymbol{\imath}} \wedge \widehat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{k}} & \widehat{\boldsymbol{\jmath}} \wedge \widehat{\boldsymbol{\imath}} = -\widehat{\boldsymbol{k}} \\ \widehat{\boldsymbol{\jmath}} \wedge \widehat{\boldsymbol{k}} = \widehat{\boldsymbol{\imath}} & \widehat{\boldsymbol{k}} \wedge \widehat{\boldsymbol{\jmath}} = -\widehat{\boldsymbol{\imath}} \\ \widehat{\boldsymbol{k}} \wedge \widehat{\boldsymbol{\imath}} = \widehat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{\imath}} \wedge \widehat{\boldsymbol{k}} = -\widehat{\boldsymbol{\jmath}}\end{array}$$

Thus we can find the component form for the vector product. For the vectors:

$$\boldsymbol{a} = a_1\widehat{\boldsymbol{\imath}} + a_2\widehat{\boldsymbol{\jmath}} + a_3\widehat{\boldsymbol{k}}$$

$$\boldsymbol{b} = b_1\widehat{\boldsymbol{\imath}} + b_2\widehat{\boldsymbol{\jmath}} + b_3\widehat{\boldsymbol{k}}$$

we find:

$$\begin{array}{rcl}\boldsymbol{a} \wedge \boldsymbol{b} & = & (a_1\widehat{\boldsymbol{\imath}} + a_2\widehat{\boldsymbol{\jmath}} + a_3\widehat{\boldsymbol{k}}) \wedge (b_1\widehat{\boldsymbol{\imath}} + b_2\widehat{\boldsymbol{\jmath}} + b_3\widehat{\boldsymbol{k}}) \\ & = & a_1(b_1\widehat{\boldsymbol{\imath}} \wedge \widehat{\boldsymbol{\imath}} + b_2\widehat{\boldsymbol{\imath}} \wedge \widehat{\boldsymbol{\jmath}} + b_3\widehat{\boldsymbol{\imath}} \wedge \widehat{\boldsymbol{k}}) \\ & & +a_2(b_1\widehat{\boldsymbol{\jmath}} \wedge \widehat{\boldsymbol{\imath}} + b_2\widehat{\boldsymbol{\jmath}} \wedge \widehat{\boldsymbol{\jmath}} + b_3\widehat{\boldsymbol{\jmath}} \wedge \widehat{\boldsymbol{k}}) \\ & & +a_3(b_1\widehat{\boldsymbol{k}} \wedge \widehat{\boldsymbol{\imath}} + b_2\widehat{\boldsymbol{k}} \wedge \widehat{\boldsymbol{\jmath}} + b_3\widehat{\boldsymbol{k}} \wedge \widehat{\boldsymbol{k}}) \\ & = & a_1b_2\widehat{\boldsymbol{k}} - a_1b_3\widehat{\boldsymbol{\jmath}} \\ & & -a_2b_1\widehat{\boldsymbol{k}} + a_2b_3\widehat{\boldsymbol{\imath}} \\ & & a_3b_1\widehat{\boldsymbol{\jmath}} - a_3b2\widehat{\boldsymbol{\imath}}\end{array}$$

So, finally:

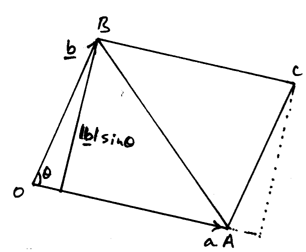
$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_2b_3 - a_3b_2)\widehat{\boldsymbol{\imath}} + (a_3b_1 - a_1b_3)\widehat{\boldsymbol{\jmath}} + (a_1b_2 - a_2b_1)\widehat{\boldsymbol{k}}$$

If you are familiar with determinants, there is an alternative way to remember this formula:

$$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{vmatrix} \widehat{\boldsymbol{\imath}} & \widehat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example

Area of a Triangle



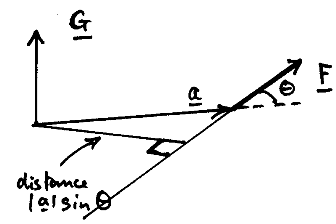
Consider a parallelogram $OACB$, with an interior angle θ between OA and OB. If $\vec{OA} = \boldsymbol{a}$ and $\vec{OB} = \boldsymbol{b}$, then the area of the parallelogram is

$$|\boldsymbol{a}|\ |\boldsymbol{b}|\ \sin\theta = |\boldsymbol{a} \wedge \boldsymbol{b}|$$

It follows that the area of the triangle OAB is

$$A = \frac{1}{2}|\boldsymbol{a} \wedge \boldsymbol{b}|$$

Moment of a Force

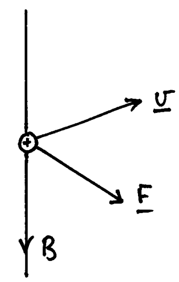


If a force $\boldsymbol{F}$ acts at some point given by position vector $\boldsymbol{a}$, then the moment of the force about the origin is defined as

$$\boldsymbol{G} = \boldsymbol{a} \wedge \boldsymbol{F}$$

The magnitude $|\boldsymbol{G}|$ is independent of the choice of the point $\boldsymbol{a}$ on the line of action of the force. (The moment depends on the perpendicular distance from the line of action of the force to the origin.)

Force on a charged particle in a magnetic field



A particle of charge q with velocity $\boldsymbol{v}$ in a magnetic field of strength $\boldsymbol{B}$ is subject to a force:

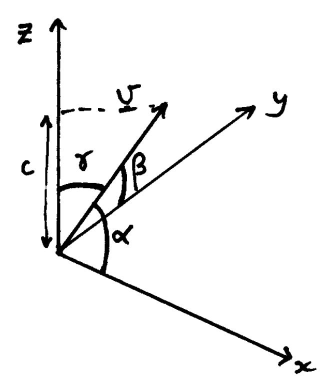
$$\boldsymbol{F} = q\boldsymbol{v} \wedge \boldsymbol{B}$$

So the force is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{B}$. A consequence is that the magnetic force cannot do any work on the particle. This force equation is used in deriving the Hall effect for charges in a semiconductor.

1.8 Equation of a Straight Line Using Vectors

A line can be thought of as all the points which lie is a certain direction from a given fixed point.

Specifying a direction in two dimensions needs just one angle, but in three dimensions it requires two. We will firt look at the relation between the components of a vector and these angles, and then move on to how to write down a vector equation for a line.



1.8.1 Direction Cosines

Consider the following vector in component form:

$$\boldsymbol{v} = a\widehat{\boldsymbol{\imath}} + b\widehat{\boldsymbol{\jmath}} + c\widehat{\boldsymbol{k}}$$

The angle between the x axis and the vector $\boldsymbol{v}$ is α; the angle between y axis and $\boldsymbol{v}$ is β; the angle between z axis and $\boldsymbol{v}$ is γ.

From the diagram, we can see that we can define a quantity n:

$$n = \cos\gamma = \frac{c}{|\boldsymbol{v}|}$$

Similarly for quantities m and l:

$$m = \cos\beta = \frac{b}{|\boldsymbol{v}|}$$

$$l = \cos\alpha = \frac{a}{|\boldsymbol{v}|}$$

The quantities l, m, and n are called the direction cosines of the vector.

Note that

$$l^2 + m^2 + n^2 = \frac{a^2 + b^2 + c^2}{|\boldsymbol{v}|^2} = 1$$

Thus the vector

$$l\hat{\boldsymbol{\imath}} + m\hat{\boldsymbol{\jmath}} + n\hat{\boldsymbol{k}}$$

is the unit vector in the direction of $\boldsymbol{v}$.

Example

$$\boldsymbol{v} = 3\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}} \quad \Rightarrow |\boldsymbol{v}| = \sqrt{3^2 + (-2)^2 + 6^2} = 7$$

Therefore:

$$l = \cos\alpha = \frac{3}{7}, \quad m = \cos\beta = \frac{-2}{7}, \quad n = \cos\gamma = \frac{6}{7}$$

⇓ .... *Hour 6* ⇓

1.8.2 Vector Equation of Line

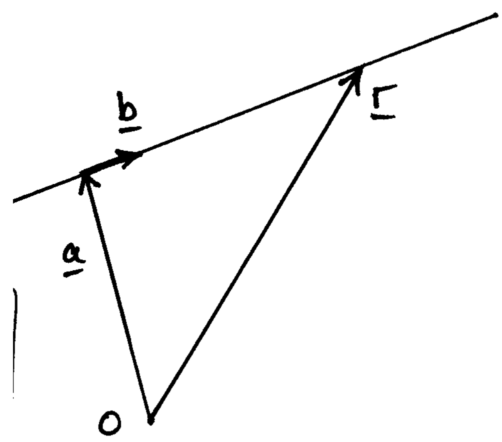
Consider this diagram:

Vector $\boldsymbol{a}$ is a position vector of a point on a line, and vector $\boldsymbol{b}$ is parallel to the line.

Then, *any* point of the line can be reached by moving to the point $\boldsymbol{a}$ on the line, and then moving by some appropriate amount parallel to the line. In other words, for an arbitrary point on the line with position vector $\boldsymbol{r}$, we must have

$$\boldsymbol{r} = \boldsymbol{a} + s\boldsymbol{b}$$

where s is a scalar parameter (i.e., some number which selects a particular $\boldsymbol{r}$. This is the vector equation of a line.



Equation of Line in Cartesian Components

If the position vector of an arbitrary point on the line is given in (Cartesian) component form, so that the point has coordinates (x, y, z):

$$\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

And, similarly the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given in component form:

$$\boldsymbol{a} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$$
$$\boldsymbol{b} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$$

Then the vector equation of the line

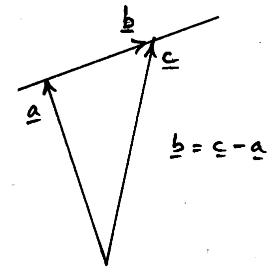
$$\boldsymbol{r} = \boldsymbol{a} + s\boldsymbol{b}$$

implies equality of the components of lhs and rhs:

$$x = a_1 + sb_1, \quad y = a_2 + sb_2, \quad z = a_3 + sb_3$$

Each of these equations can be solved for the number s:

$$s = \frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$



This is the *equation of the line in Cartesian coordinates.*

Special case: If $\boldsymbol{b}$ is a unit vector, then b_1, b_2, b_3 are the direction cosines of the line.

Special case: If the line passes through the origin, then $\boldsymbol{a} = \boldsymbol{0}$ and $a_1 = a_2 = a_3 = 0$.

If, instead, two position vectors, $\boldsymbol{a}$ and $\boldsymbol{c}$, are given of points on the line, then, by the vector addition law we can find a vector parallel to the line:

$$\boldsymbol{b} = \boldsymbol{c} - \boldsymbol{a}$$

so that the equation of the line, in this case, becomes

$$\boldsymbol{r} = \boldsymbol{a} + s(\boldsymbol{c} - \boldsymbol{a})$$

Example

Find the equation of the straight line which makes an angle of 45° to the z axis, and an angle of 60° to the x axis, and which passes through the origin. (Note that two angles are required to specify the direction of the line.)

The direction cosines l (cosine of angle to x axis) and n (cosine of angle to z axis) are obvious:

$$l = \cos\alpha = \cos 60^\circ = \frac{1}{2}$$

$$n = \cos\gamma = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

The third direction cosine is found from the relationship: $l^2 + m^2 + n^2 = 1$, so

$$m = \sqrt{1 - l^2 - n^2} = \sqrt{1 - \frac{1}{4} - \frac{1}{2}} = \frac{1}{2}$$

The line passes through the origin, so the vector $\boldsymbol{a}$ (as in previous section) is zero: $\boldsymbol{a} = 0$. And the vector

$$l\widehat{\boldsymbol{\imath}} + m\widehat{\boldsymbol{\jmath}} + n\widehat{\boldsymbol{k}}$$

is parallel to the line.

So the equation of the line is:

$$\boldsymbol{r} = s(\frac{1}{2}\widehat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\widehat{\boldsymbol{\jmath}} + \frac{1}{2}\widehat{\boldsymbol{k}})$$

where s is parameter (each point on the line corresponds to a different value for s).

The Cartesian (i.e., xyz) form for the equation of the line can be found by noting that

$$\boldsymbol{r} = x\widehat{\boldsymbol{\imath}} + y\widehat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}$$

where x, y and z are the coordinates of a point on the line, and they satisfy:

$$\frac{x}{l} = \frac{y}{m} = \frac{x}{n}$$

So that, x, y, and z are related in the ratios:

$$2x = 2y = \sqrt{2}\, z$$

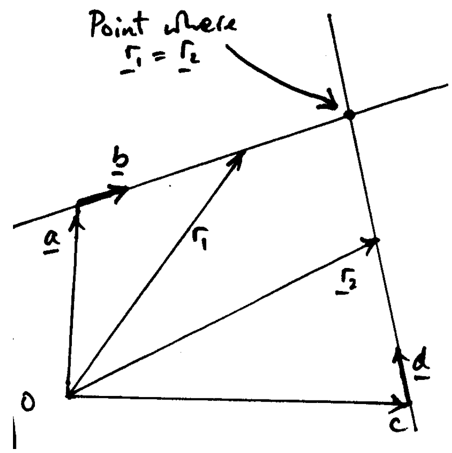
1.8.3 Intersection of Two Lines

Suppose that there are two lines given, respectively, by the two equations:

$$\boldsymbol{r}_1 = \boldsymbol{a} + s\boldsymbol{b}, \quad \boldsymbol{r}_2 = \boldsymbol{c} + t\boldsymbol{d}$$

Where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are given vectors, and s and t are parameters.

Unlike the case for 2D (when lines always intersect unless they are parallel), there are three possibilities: (1) the lines do not intersect; (2) they intersect at one point; (3) the two lines are identical.



Let us deal with Case (3): clearly $\boldsymbol{b}$ and $\boldsymbol{d}$ must be in the same direction, or exactly opposite direction, so it follows that

$$\boldsymbol{d} = p\,\boldsymbol{b}$$

where p is some number. Also, $\boldsymbol{a}$ and $\boldsymbol{c}$ must both be position vectors for points on the line, so $(\boldsymbol{a} - \boldsymbol{c})$ must be in the direction of the line, and it follows that

$$(\boldsymbol{a} - \boldsymbol{c}) = q\,\boldsymbol{b}$$

If these two conditions are satisfied, then the two lines are identical.

To look for possible intersection of the lines, there must exist a point which satisfies *both* equations:

$$\boldsymbol{r} = \boldsymbol{a} + s^*\,\boldsymbol{b} = \boldsymbol{c} + t^*\,\boldsymbol{d}$$

where s^* and t^* are the particular values for the intersection point. In Cartesian component form this gives *three* equations for the *two* unknowns (namely s^* and t^*):

$$a_1 + s^*\,b_1 = c_1 + t^*\,d_1$$
$$a_2 + s^*\,b_2 = c_2 + t^*\,d_2$$
$$a_3 + s^*\,b_3 = c_3 + t^*\,d_3$$

Two of these equations can be used to find values for s^* and t^*, but the lines will only truly intersect if the third equation is also satisfied for the same values of s^* and t^*. Thus it is important to check for intersection in all *three* components.

Example

Let us look at two simple examples. Consider two lines whose vector equations are

$$\begin{aligned} \boldsymbol{r}_1 &= (1,0,0) + s(-1,1,0) \\ \boldsymbol{r}_2 &= t(1,1,1) \end{aligned}$$

We are using the short hand notation $(1,1,1) = \widehat{\boldsymbol{\imath}} + \widehat{\boldsymbol{\jmath}} + \widehat{\boldsymbol{k}}$. Note how different parameters s and t are used for the different lines.

The aim is to find values for s **and** t such that $\boldsymbol{r}_1\ \boldsymbol{r}_2$. This condition leads to three simultaneous equations:

$$1 - s = t$$

$$\begin{aligned} s &= t \\ 0 &= t \end{aligned}$$

The last two equations lead to $s = t = 0$, but this is clearly inconsistent with the first equation. So, in this case **the lines do not intersect.**

Next consider:

$$\begin{aligned} \boldsymbol{r}_3 &= (1,0,0) + s(-1,1,1) \\ \boldsymbol{r}_2 &= t(1,1,1) \end{aligned}$$

Similarly, the simultaneous equations to satisfy are:

$$\begin{aligned} 1 - s &= t \\ s &= t \\ s &= t \end{aligned}$$

The last two equations are now identical, so we can solve the first:

$$1 - t = t \quad \Rightarrow \quad t = 1/2, \quad s = 1/2$$

so, in this case, **the lines do intersect.** And by substituting into $\boldsymbol{r}_2$ (or $\boldsymbol{r}_3$) we find the point of intersection:

$$\boldsymbol{r} = (1/2)(1,1,1) = \frac{1}{2}(\widehat{\boldsymbol{\imath}} + \widehat{\boldsymbol{\jmath}} + \widehat{\boldsymbol{k}})$$
