MAE111 Engineering Mathematics II
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SECTION 4: SERIES

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Section 4

Sequences and Series

In this chapter we will see how we can use what we know about differentiation to approximate almost any functionof x by a series of terms involving just powers of x. We will also learn how to test whether a sum with an infinite number of terms can actually converge to a single, finite result.

4.1 Maclaurin's Series

It is possible to show that any (well, almost any) function can be approximated as a power series in x. In other words, there is a power series which is an equivalent way of expressing the function.

4.1.1 Polynomial Functions

We are familiar with linear functions:

$$f(x) = a_0 + a_1 x$$

where a_0 and a_1 are constant. We are equally familiar with quadratic functions such as:

$$f(x) = a_0 + a_1 x + a_2 x^2$$

. . . *and so on.*

Clearly there is a pattern here that we can generalise. A function which is a sum of a number of terms where each term depends *only* on integral powers of x is called a polynomial function.

Consider a polynomial function with terms up to x^n (called an n-th order polynomial):

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

where the a_n are the coefficients of the polynomial. A polynomial is an example of a **power series**.

Some Definitions

In order to take our discussion further we need to make some definitions.

A **sequence** is an ordered set of quantities

$$u_1, u_2, u_3, \ldots$$

such as 2,4,6,8, or what ever. Usually there is some pattern in the terms, but a sequence might be a set of random numbers. Note how we are using subscripts to label the different terms, ie first, second, third, etc.

A sequence can be **finite**, e.g., the even integers up to 100:

$$2, 4, 6, \ldots, 100$$

or **infinite**, in which case it continues indefinitely e.g., sequence formed, starting from 2 and repeatedly doubling:

$$2, 4, 8, 16, 32, 64, 128, \ldots$$

A **series** is the sum of the terms of a sequence. The sum of n terms can be written

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{r=1}^{n} u_r$$

For finite value of n, then S_n is a **finite series**.

If the number of terms is infinite (i.e., $n \to \infty$) then the sum is an **inifinite series** and one writes

$$S_\infty = u_1 + u_2 + u_3 + u_4 + u_5 + \ldots = \sum_{r=1}^{\infty} u_r$$

A **power series** is one particular case where the terms only depend on powers of a variable, such as x.

One can have power series in x which are either infinite:

$$S = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \quad S_\infty = \sum_{n=0}^{\infty} a_n x^n$$

or finite:

$$S = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_N x^N \quad S_N = \sum_{n=0}^{N} a_n x^n$$

Since the terms in the series now depend on a variable, the whole series must be considered as a function of x: $S \equiv S(x)$.

Polynomial Coefficients

There is a striking relationship between the coefficients of a polynomial and the derivatives of the function. We can find the coefficients in terms of $f(x)$ and its derivatives, by repeatedly differentiating and setting $x = 0$.

First the result of repeated differentiation:

$$\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n \\
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{(n-1)} \\
f''(x) &= 2a_2 + 3 \cdot 2a_3 x + \ldots + n(n-1) a_n x^{(n-2)} \\
f'''(x) &= 3 \cdot 2a_3 + \ldots + n(n-1)(n-2) a_n x^{(n-3)}
\end{aligned}$$

Then set $x = 0$:

$$\begin{aligned} a_0 &= f(0) \\ a_1 &= f'(0) \\ a_2 &= \frac{1}{2}f''(0) \\ a_3 &= \frac{1}{3!}f'''(0) \end{aligned}$$

Note that $f'(0)$ means the first derivative of f evaluated at $x = 0$ and we have used the factorial function

$$n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1.$$

Thus for the polynomial we can write a power series version of the original function in terms of the derivatives of $f(x)$.

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \ldots + \frac{1}{n!}f^{(n)}(0)x^n$$

Now, a polynomial obviously is just a finite power series, so we have simply found a relationship between the coefficients of the polynomial and the derivatives of the polynomial function.

4.1.2 Approximating a Function

Of course, so far we have found a connection between differentiating the function and it coefficients, but we haven't really learnt anything new!

But, what about functions which are *not* polynomial? Might it be possible to *approximate* such a function by a polynomial?

For example, a function close to some point might be thought of as roughly like a straight line (ie the tangent line) which is a linear function. If we zoom in closer to the point we might see that the function isn't quite a straight line but is something like a quadratic. And if we zoom in again, it might look like a cubic … *and so on!*

However, since we are looking at an approximation to a function, that is not itself polynomial, we should expect there to be some amount of difference between the function itself and its power series reprepresentation. This difference is known as the *Remainder term*.

4.1.3 Maclaurin's Series

We now come to the statement of what is known as Maclaurin's theorem:

It is possible to write down a power series representation of a function, together with a remainder term. The remainder term must tend to zero as the number of terms in the series increases, so that the power series is an ever more close approximation to the function.

The terms in the power series are just those detailed above, which implies that the function must be able to be differentiated the appropriate number of times. (For most of the functions we are interested in, they can be differentiated any number of times.)

Maclaurin's Series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + R_n(x)$$

where the remainder term is

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

where ξ is a number such that $0 \leq \xi \leq x$ (for $x > 0$).

There are some points to note, that could mean that the Maclaurin's series cannot be used: As already mentioned, the remainder term must get smaller as n increases, so that the approximation improves as more terms are added. The function $f(x)$ might not be differentiable, or only differentiable for some range of x, and in that case the series cannot be constructed, or is only valid over some range in x.

If the function $f(x)$ can be differentiated an infinite number of times, then the Maclaurin's Series becomes an infinite power series:

$$f(x) = \sum_{0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

but only if it can be shown that this series converges (for example, by using the ratio test) (this will be discussed shortly!).

Once we know that we can use a Maclaurin's series as an approximation for a function, it becomes a very powerful tool, because it means that we can calculate the value of the function using only the standard rules of arithmetic. This is especially important for calculations using computers.

Example

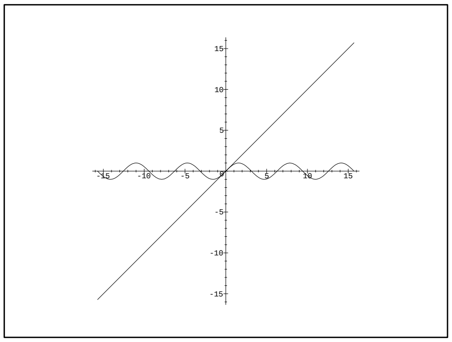
Maclaurin's series for $f(x) = sin(x)$

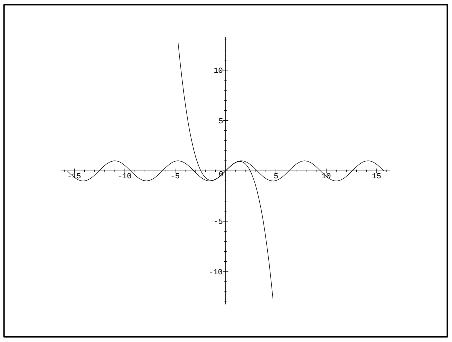
$$\begin{array}{lllll} f = \sin x & f' = \cos x & f'' = -\sin x & f''' = -\cos x & \dots \\ f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f'''(0) = -1 & \dots \end{array}$$

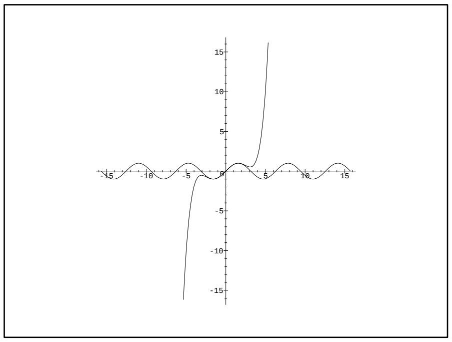
So,

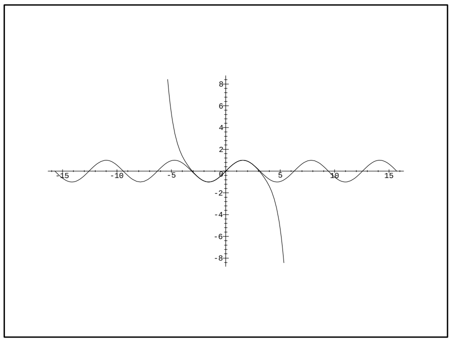
$$\begin{aligned} \sin x &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + ... \\ \sin x &= \sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$

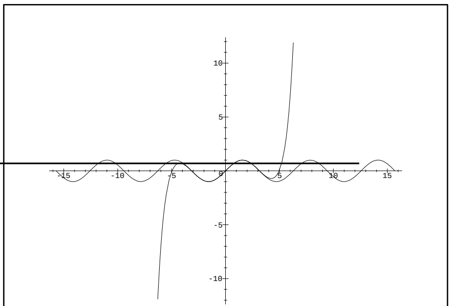
As we will see later, this result is correct for all values of x.











Example

Let us consider the Maclaurin's series for the function $f(x) = e^x$:

$$f' = e^x, \quad f''(x) = e^x, \quad f''' = e^x,$$

So,

$$f(0) = 1, \quad f''(0) = 1, \quad f''' = 1, \quad \text{etc.}$$

And it follows that we can write down the infinite series for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

As we shall see shortly this series converges for all values of x, and is none other than the well known series for the exponential function.

Example

We can find the Maclaurin's series for $f(x) = \cos x$ in a similar way, but here we will find it by *differentiating* the series for $\sin x$. (The derivative of a series can be found by differentiating term by term.) Then,

$$\cos x = \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Example

Maclaurin's series for $\cosh x$:

$$\begin{array}{lllll} f = \cosh x & f' = \sinh x & f'' = \cosh x & f''' = \sinh x & \ldots \\ f(0) = 1 & f'(0) = 0 & f''(0) = 1 & f'''(0) = 0 & \ldots \end{array}$$

So,

$$\begin{aligned} \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... \\ \cosh x &= \sum_{0}^{\infty} \frac{x^{2n}}{(2n)!} \end{aligned}$$

Example

Maclaurin's series for $\ln(1 + x)$:

$$\begin{array}{lllll} f = \ln(1+x) & f' = \frac{1}{x+1} & f'' = -(1+x)^{-2} & f''' = 2(1+x)^{-3} & \ldots \\ f(0) = 0 & f'(0) = 1 & f''(0) = -1 & f'''(0) = 2 & \ldots \end{array}$$

So,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} ...$$

In this case, one can show that the series is only correct (ie will only converge) for $|x| \leq 1$. This is called conditional convergence.

A Problem?

Although the Maclaurin's Series method seems incredibly powerful and useful (it means that we can calculate functions approximately just using the standard arithmetic operations), there seems to be a problem.

Surely adding up an *infinite* number of terms is going to lead to a value that is itself infinite? On the other hand, if the size of the terms gets smaller and smaller, then perhaps we can imagine that adding more and more eventually makes less and less difference to the answer. In other words, how can we be sure that the Maclaurin's Series actually produces the right answer?

To answer this question we need to look at sequences and series in some more detail.

4.2 Sequences and Series in General

A **sequence** is an ordered set of quantities

$$u_1, u_2, u_3, \ldots$$

such as 2,4,6,8, or what ever. Usually there is some pattern in the terms, but a sequence might be a set of random numbers.

A sequence can be **finite**, e.g., the even integers up to 100:

$$2, 4, 6, \ldots, 100$$

or **infinite**, in which case it continues indefinitely e.g., sequence formed, starting from 2 and repeatedly doubling:

$$2, 4, 8, 16, 32, 64, 128, \ldots$$

A **series** is the sum of the terms of a sequence. The sum of n terms can be written

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{r=1}^{n} u_r$$

For finite value of n, then S_n is a **finite series**.

If the number of terms is infinite (i.e., $n \to \infty$) then the sum is an **inifinite series** and one writes

$$S_\infty = u_1 + u_2 + u_3 + u_4 + u_5 + \ldots = \sum_{r=1}^{\infty} u_r$$

4.2.1 Arithmetic series

If the difference between subsequent terms is a constant amount d, then the series is arithmetic, and the terms are:

$$u_1 = a, \quad u_2 = a + d \quad u_p = u_{p-1} + d = a + (p-1)d$$

Then the series can be written

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) + (a+3d) + \ldots + (a+(n-1)d) \\ &= na + (1 + 2 + 3 + \ldots + (n-1))d \\ &= na + \frac{1}{2}n(n-1)d \end{aligned}$$

Where we are using the well known result that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Example

For $a = 2$, $d = 3$:

$$S = 2 + 5 + 8 + \ldots$$

And so

$$S_{10} = 10.(2) + \frac{1}{2}10\ .\ 9\ .\ 3 = 20 + 135 = 155$$

4.2.2 Geometric Series

If the terms, starting from a, differ by a constant multiplicative factor r, then the series is geometric, and the terms are:

$$u_1 = a \quad u_2 = ar \quad u_3 = ar^2 \quad u_p = ar^{p-1}$$

And the series of n terms is

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \\ &= a \sum_{p=0}^{n-1} r^p \end{aligned}$$

This can be put in closed form, by noting:

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \\ rS_n &= \phantom{a +{}} ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \\ S_n - rS_n &= a - ar^n \end{aligned}$$

So that:

$$S_n = \frac{a(1-r^n)}{1-r}$$

This is the best form for $|r| < 1$. If, on the other hand, $|r| > 1$ then use

$$S_n = \frac{a(r^n - 1)}{r-1}$$

Example

Consider geometric series with $a = 8$ and $r = \frac{1}{2}$

$$S_n = 8 + 4 + 2 + \ldots$$

So,

$$S_8 = 8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

And from formula

$$S_8 = 8\frac{\left(1 - \frac{1}{2^8}\right)}{1 - \frac{1}{2}} = 8\frac{\frac{255}{256}}{\frac{1}{2}} = \frac{255}{16}$$

4.3 Convergence and Divergence

We now consider what can happen when we start to sum an infinite series. One can obtain an infinite series from a finite series by allowing the number of terms included in the sum to increase towards infinity. That is we consider the series S_n (sometimes called the partial sum) and let $n \to \infty$.

A series is **convergent** if the quantity

$$\lim_{n\to\infty} S_n$$

exists, and has a finite value.

The contrary case can easily be imagined: if we have a series whose terms are all the same sign, but which get bigger and bigger, then the more terms that are summed will makes the series sum even larger.

A series is **divergent** if the quantity

$$\lim_{n\to\infty} S_n$$

does not tend to some definite value, for example if $\lim_{n\to\infty} S_n = \infty$. If S_n is always changing as $n \to \infty$ but remaining finite, so that it does not have a single limiting value, then the series is called divergent in this case also.

We can demonstrate examples of convergence and divergence using geometric series, and the fact that we have an explicit form of the sum of the first n terms:

$$S_n = \frac{a(1-r^n)}{1-r}.$$

As an example of convergence, consider the geometric series with $a = 1$, and $r = \frac{1}{2}$.

$$\begin{aligned} S_n &= 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}} \\ &= (1).\frac{\left(1 - \frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)} = 2\left(1 - \frac{1}{2^n}\right) \end{aligned}$$

Now, as $n \to \infty$, $\frac{1}{2^n} \to 0$, and so the sum tends to the value of two, which we write as

$$\lim_{n\to\infty} S_n = 2$$

This simple example, is very well behaved: Each term has the same sign and is smaller than the one before. The sum as $n \to \infty$ is a well behaved limit, ie a finite limiting value exists.

As an example of divergence, consider the geometric series with $a = 1$ and $r = 2$

$$\begin{aligned} S_n &= 1 + 4 + 2 + \ldots + 2^{n-1} \\ &= (1).\frac{(2^n - 1)}{(2-1)} = (2^n - 1) \end{aligned}$$

and as $n \to \infty$, $S_n \to \infty$, since 2^n increases without limit. Thus in this case,

$$\lim_{n\to\infty} S_n = \infty$$

and the series is divergent.

4.3.1 Summary

A series for which S_n tends to a definite finite value V as $n \to \infty$ is called a **convergent** series. If S_n does not tend to such a value, the series is called **divergent**.

For a convergent series, we can write

$$\lim_{n\to\infty} S_n = V$$

where V is the sum to infinity of the series.

4.4 Tests for Convergence

Whether or not a series converges can be proved by finding an expression for S_n, and then studying the limiting value of this sum as $n \to \infty$. This is what has just been done for the general geometric series.

However, for many series it is not possible to find an expression for S_n. In such cases, there are nevertheless tests which can be carried out to explore the convergence properties of the series. Here we will give these tests (without proofs), and examples. The important point to note is sometimes a test does not return a definite answer about the convergence of a series, and in such cases further tests have to be carried out.

We will initially consider series whose terms are positive: all $u_n > 0$.

4.4.1 "Terms non-vanishing" test

A series whose terms do not tend to zero in magnitude, i.e.,

$$\lim_{n\to\infty} u_n \neq 0$$

is certainly divergent.

But the contrary is not true: that is, one can have a series whose terms tend to zero, but which is nevertheless divergent. Or in other words: just because the terms tend to zero it is no guarantee of convergence.

This test is relatively easy to perform, and can quickly rule out convergence in some cases.

4.4.2 "Term by term comparison" test

This kind of test works by comparing, term by term, the series with another series whose convergence is known.

A series will be convergent if it has terms, which have one to one correspondence with, and are always smaller than, the terms of another series which is already known to converge.

Similarly, a series will be divergent if it has terms, which have one to one correspondence with, and are always larger than, the terms of another series which is already known to diverge.

4.4.3 "Ratio" Test

This is one of the most powerful tests.

Consider the quantity which is the limit of the (absolute) ratio of the last two terms in the series as $n \to \infty$:

$$k = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

Note we take absolute value to deal with cases where terms are negative.

If …	$k < 1$	series converges
	$k > 1$	series diverges
	$k = 1$	inconclusive

If the quantity $k = 1$ then the test gives no information on the convergence of the series. If $k = 1$ then no conclusion can be drawn!

Example

Consider:

$$S_\infty = \sum_1^\infty \frac{1}{n!}$$

Then,

$$u_n = \frac{1}{n!} \quad \frac{u_{n+1}}{u_n} = \frac{1}{n+1}$$

Thus,

$$k = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left(\frac{1}{n+1} \right) = 0$$

Therefore, the series (which is none other than e^1) converges.

For the test to work the limit

$$\lim_{n\to} \left| \frac{u_{n+1}}{u_n} \right|$$

must tend to a *single* value.

For example, consider this series:

$$u_n = \frac{(3 + (-1)^n)}{n^2}$$

The terms are alternately $4^n/n^2$ and $2^n/n^2$, so the ratio of subsequent terms is alternately (for large n)

$$\frac{4^n}{2^n}, \frac{2^n}{4^n}, \quad \text{i.e., } 2^n, \frac{1}{2^n}$$

So, for this series, the ratio test gives no conclusion.

4.5 Convergence of Geometric Series

Note: this section is not given in Lectures.

As an example of the different kinds of behaviour that series can exhibit, we will examine the convergence properties of the general geometric series

$$S_n = a \sum_{p=0}^{n-1} r^p$$

We have the results

$$\begin{aligned} S_n &= a \left(\frac{1 - r^n}{1 - r} \right) \quad \text{for } |r| \neq 1 \\ S_n &= na \quad \text{for } r = 1 \end{aligned}$$

(The second result is obvious.)

As $n \to \infty$, there are FIVE different cases to consider, depending on the value of the factor r.

Convergence for $|r| < 1$ ($-1 < r < +1$)

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, so

$$\lim_{n\to\infty} S_n = \frac{a}{1 - r}$$

which is finite. So, the series is convergent for this range of r.

Convergence for $r = +1$

If $r = +1$, then we have the simpler formula: $S_n = na$, thus clearly,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (na) = \infty$$

Since there is no finite limiting value, the series is divergent in this case.

Convergence for $r > +1$

In this case we have

$$\lim_{n\to\infty} r^n = \infty$$

so applying the formula for S_n gives

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1 - r^n}{1 - r} \right) = \infty$$

So, the series is divergent in this case.

Convergence for $r = -1$

In the case $r = -1$, the sign of the terms alternates, but the terms have the same absolute magnitude:

$$S_n = a - a + a - a + a + \dots$$

So the sum is either zero, or a, depending on whether there are an even or odd number of terms! Thus, although the series remains finite it oscillates in value, so the series is divergent for $r = -1$.

Convergence for $r < -1$

For $r < -1$, and as $n \to \infty$, the series is dominated by the behaviour of r^n which is either $+\infty$ or $-\infty$, depending if n is odd or even. Thus, as for $r = -1$, the series is divergent for $r < -1$.

Summary

A geometric series is only convergent if

$$-1 < r < +1$$

4.6 Examples of Convergence Testing

These examples show the use of the Ratio and other tests. Such tests rely on knowing how some simple limits behave, e.g., for positive p:

$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

Also, when there is a limit with positive powers of n in a fraction, and it thus has the indeterminate form $\frac{\infty}{\infty}$, then one should divide top and bottom by the highest power of n, and then take the limit. For example:

$$\lim_{n\to\infty} \frac{n^3 + 1}{3n^3 + 3n - 7} = \lim_{n\to\infty} \frac{1 + 1/n^3}{3 + 3/n^2 - 7/n^3} = \frac{1}{3}$$

4.6.1 Example 1

For series with general term:

$$u_n = \frac{n-1}{n+2}$$

Obviously:

$$u_{n+1} = \frac{n}{n+3}$$

So,

$$\frac{u_{n+1}}{u_n} = \frac{n}{n+3} . \frac{n+2}{n-1} = \frac{n^2+2n}{n^2+2n-3}$$

Then for the Ratio test:

$$k = \lim_{n\to\infty} \frac{n^2+2n}{n^2+2n-3}$$

In its present form it is an indeterminate form (∞/∞), so divide top and bottom by largest power of n (i.e., n^2):

$$k = \lim_{n\to\infty} \frac{1+2/n}{1+2/n-3/n^2} = 1$$

Since $k = 1$ the Ratio test is <u>inconclusive</u>.

But we can use the "terms non-vanishing" test:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n-1}{n+2} = \lim_{n\to\infty} \frac{1-1/n}{1+2/n} = 1$$

And since the terms do not vanish away as $n \to \infty$, therefore the series <u>diverges</u>.

The Ratio test is usually done first, since it gives the chance that convergence of the series will be proved immediately. Of course, there is also the chance that the Ratio test will prove inconclusive.

4.6.2 Example 2

For series with general term:

$$u_n = \frac{3^n}{n!}, \quad \Rightarrow u_{n+1} = \frac{3^{n+1}}{(n+1)!}$$

So the ratio of successive terms is

$$\frac{u_{n+1}}{u_n} = \frac{3^{n+1}}{(n+1)!} . \frac{n!}{3^n} = \frac{3}{n+1}$$

Then

$$k = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 3 \lim_{n\to\infty} \frac{1}{n+1} = 0$$

And, since $k < 1$, the series <u>converges</u>.

4.6.3 Example 3

For series with general term:

$$u_n = \frac{(n-1)!}{4^n(n^2+1)}, \quad \Rightarrow u_{n+1} = \frac{n!}{4^{n+1}\left[(n+1)^2+1\right]}$$

The ratio of successive terms is:

$$\frac{u_{n+1}}{u_n} = \frac{n!}{4^{n+1}\left[(n+1)^2+1\right]} \cdot \frac{4^n(n^2+1)}{(n-1)!} = \frac{n}{4}\frac{(n^2+1)}{\left[(n+1)^2+1\right]}$$

Then, dividing top and bottom by n^2

$$k = \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{1}{4}\lim_{n\to\infty}\frac{n(1+1/n^2)}{\left[(1+1/n)^2+1/n^2\right]} = \frac{1}{4}\lim_{n\to\infty} n = \infty$$

and since $k > 1$ (obviously!), the series is divergent.

4.7 Convergence of Power Series

There is one particular case of a series where the terms involve the powers of a variable, such as x. One can have, as before, power series in x which are either infinite:

$$S = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \quad S_\infty = \sum_{n=0}^{\infty} a_n x^n$$

or finite:

$$S = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_N x^N \quad S_N = \sum_{n=0}^{N} a_n x^n$$

Since the terms in the series now depend on a variable, the whole series must be considered as a function of x: $S \equiv S(x)$.

For power series the question of convergence or divergence is vital. For an inifinite power series one can try the Ratio test (assuming that the suitable limit actually exists). Then the following condition:

$$k = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}x\right| < 1$$

must be true in order for the series to converge. This can be manipulated to give a condition on the variable x:

$$|x| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

In other words, unless the RHS is infinite, the series will *only* converge for a certain range of x values. The quantity on the RHS of the above inequality is somtimes called the *radius of convergence* for the series.

Example

Consider the series for e^x

$$S = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_0^\infty \frac{x^n}{n!}$$

This series will converge for

$$|x| < \lim_{n\to\infty}\left(\frac{1/n!}{1/(n+1)!}\right) = \lim_{n\to\infty}(n+1) = \infty$$

So, the condition for convergence is that $|x|$ must be less than infinity, which is always true, so the series will converge for all (finite) values of x.

Example

Convergence of Maclaurin's Series for $\sin(x)$

The power series for $\sin(x)$ is:

$$\begin{aligned}\sin x &= 0+x+0-\frac{x^3}{3!}+0+\frac{x^5}{5!}+...\\ \sin x &= \sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}\end{aligned}$$

We now need to make sure that this series actually converges! Using the ratio test:

$$k=\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right| = 0$$

This follows, since the limit varies as n^{-2}, which tends to zero as n tends to infinity. Since the limit is always zero, and $k < 1$ for convergence, then the series for $\sin x$ will converge for all values of x.

4.7.1 The Binomial Series

The Binomial expansion is well known:

$$\begin{aligned}(1+x)^p &= 1+px+\frac{p(p-1)}{2!}x^2+\dots\\ &= \sum_{n=0}\frac{p!x^n}{(p-n)!n!}\end{aligned}$$

The most familiar case is when p is a positive integer, in which case the $(p+1)$th term is zero, as are all the subsequent terms. So, for integral p the binomial expansion produces a finite power series.

$$\begin{aligned}(1+x)^0 &= 1\\ (1+x)^1 &= 1+x\\ (1+x)^2 &= 1+2x+x^2\\ (1+x)^3 &= 1+3x+3x^2+x^3\\ (1+x)^4 &= 1+4x+6x^2+4x^3+x^4\end{aligned}$$

Notice how the Binomial expansion produces the coefficients from Pascal' triangle.

However, one major application of Maclaurin's series, is to show that the Binomial series is true, even when p is *not* an integer, in which case the series is infinite (and convergence has to be checked).

$$\begin{array}{lllll} f=(1+x)^p & f'=p(1+x)^{p-1} & f''=p(p-1)(1+x)^{p-2} & f'''=p(p-1)(p-2)(1+x)^{p-3} & \dots\\ f(0)=1 & f'(0)=p & f''(0)=p(p-1) & f'''(0)=p(p-1)(p-2) & \dots\end{array}$$

From this it is clear that the Maclaurin's series for $(1+x)^p$ is just the Binomial expansion. And since the Maclaurin's series is valid for p not being an integer, then so too must the Binomial expansion.

In the case where the series is infinite, we use the ratio test for convergence, i.e., the following must be true:

$$k=\lim_{n\to\infty}\left|\frac{\frac{p!x^{n+1}}{(p-n-1)!(n+1)!}}{\frac{p!x^n}{(p-n)!n!}}\right| = \lim_{n\to\infty}\left|\frac{x(p-n)}{(n+1)}\right| = |x|\lim_{n\to\infty}\left|\frac{p}{n}-\frac{n}{n+1}\right| < 1$$

So the condition for convergence of an infinite Binomial series is

$$|x| < 1$$

Example

The Binomial series expansion is immensely useful, and we can use it to demonstrate another way in which series may be manipulated.

We wish to find the series expansion for $\tan^{-1} x$, and as a trick, we first differentiate!

$$f(x) = \tan^{-1} x \quad \Rightarrow f'(x) = (1 + x^2)^{-1}$$

Now, a series for the derivative follows straightaway from the Binomial expansion:

$$\begin{aligned} f'(x) &= 1 - x^2 + \frac{(-1)(-2)}{1 \cdot 2} x^4 + \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3} x^6 + \dots \\ &= 1 - x^2 + x^4 - x^6 + \dots = \sum_0^\infty (-1)^n n x^{2n} \end{aligned}$$

Now, we *integrate* both sides of this equation (we can integrate the power series term by term), to find

$$\tan^{-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

Example

Calculation of electrostatic potential of two opposite charges.

Consider two charges $\pm Q$ at points B and A, respectively. This is called an **electric dipole**. The distance AB is $2s$. We want to calculate the combined potential of the two charges at a point C which is a distance r from the midpoint of AB in the direction of AB. In other words we are only going to calculate the potential along the axis of the dipole, which simplifies the calculation.

The electric potential at C is

$$V = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r+s} - \frac{1}{r-s} \right).$$

Writing

$$\frac{1}{r+s} = (r+s)^{-1} = r^{-1} \left(1 + \frac{s}{r} \right)^{-1}$$

and similarly

$$\frac{1}{r-s} = (r-s)^{-1} = r^{-1} \left(1 - \frac{s}{r} \right)^{-1}.$$

We are interested in the situation when $r \gg s$, ie $s/r \ll 1$. So we use two useful Binomical expansions, both valid for $x < 1$:

$$\begin{aligned} (1+x)^{-1} &= 1 - x + x^2 - x^3 + \dots \\ (1-x)^{-1} &= 1 + x + x^2 + x^3 + \dots \end{aligned}$$

Using these to expand V in terms of s/r, and only keeping terms up to $(s/r)^2$, gives

$$\begin{aligned} V &= \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \left[1 - \frac{s}{r} + \left(\frac{s}{r} \right)^2 - \left(1 + \frac{s}{r} + \left(\frac{s}{r} \right)^2 \right) \right] \\ &= \frac{Q}{4\pi\epsilon_0} \frac{2s}{r^2} \end{aligned}$$

Thus our final result is that the potential of the dipole, along its axis, varies as $1/r^2$, instead $1/r$ as for a single charge.

4.8 Taylor's Theorem

Now, Maclaurin's series uses the information about $f, f', f'', f''', \ldots$ (i.e. the function and all its derivatives) *evaluated at* $x = 0$ to "reconstruct" the function f as a power series. The obvious question is: What is special about zero? Indeed, there is nothing special about the choice of zero, other than that it ensures that the individual components can be identified.

We will now show that Maclaurin's series is only a special case of a more general series, known as Taylor's series. (Taylor was a student of Newton, and, apparently, there is little historical justification to associate Maclaurin's name with the Maclaurin's series!)

Consider Maclaurin's series for the function $g(X)$:

$$g(X) = g(0) + Xg'(0) + \frac{X^2}{2!}g''(X) + \ldots$$

Now, let $X = x - a$, where a is some constant.

$$g(x-a) = g(0) + (x-a)g'(0) + \frac{(x-a)^2}{2!}g''(X) + \ldots$$

If $X = 0$ then it implies that $x = a$. We can define a new function of x

$$f(x) = g(x-a) \quad \Rightarrow f'(x) = g'(x-a)$$

which is just the function g shifted by an amount a. Then we can write

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$$

This is the Taylor's series for $f(x)$ about the position $x = a$.

An alternative form is to express the series in terms of the distance of x from a, i.e., to introduce $h = x - a$, and then to replace a by x, so that the function and its derivatives are evaluated at x:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots$$

This can be thought of as a way of calculating f at $(x + h)$, when given f and its derivatives at x. If h is sufficiently small, then only keeping a few terms of the power series is a way of approximating $f(x + h)$ in the vicinity of x. The approximation is more accurate when h is small, i.e., closer to the point where the function and its derivatives are given.

Taylor's series has innumerable applications in numerical methods (where approximations to the derivatives are required) for approximate computation of functions, and for estimating errors in, for example, measurements.

4.8.1 Function Approximation by Taylor's Series

Sometimes it is useful to have a linear or quadratic approximation to a function, for example to simplify the solution of equations involving the function.

Example

As an example we can consider the current (I) – voltage (V) characteristic of a diode (at room temperature)

$$I(V) = I_s(e^{40V} - 1)$$

I is the diode current, V the applied voltage, and I_s is the constant reverse saturation current.

Now, if the diode is operating close to a voltage value V_a, then the second order Taylor polynomial (ie quadratic) is

$$I_{p2}(V) = I(V_a) + (V - V_a)I'(V_a) + \frac{(V - V_a)^2}{2} I''(V_a)$$

where

$$I'(V) = 40I_s e^{40V}, \quad I''(V) = 1600 I_s e^{40V}$$

so

$$I_{p2} = I_s(e^{40V_a} - 1) + 40I_s e^{40V_a}(V - V_a) + 800 I_s e^{40V_a}(V - V_a)^2$$

The coefficients in this quadratic only need to be computed once. In some situations, the use of a quadratic makes the computation of the current much easier, or perhaps simplifies the solution of V in terms of I. Of course, this quadratic relationship is only approximate, and care must be taken to only use it where the error in the approximation is sufficiently small.

One particular use is to consider only the linear term at some operating voltage V_a, which allows us to calculate the so-called slope resistance at this operating point, by considering the change in current associated with a chagne in V:

$$\frac{\Delta V}{\Delta I} = (40 I_s e^{40V_a})^{-1} = \frac{1}{40 I_a},$$

where we have used the current at the operating point:

$$I_a = I_s(e^{40V_a} - 1) \approx I_s e^{40V_a}.$$

4.8.2 L'Hopital's Rule

This section deals with a useful application of Taylor's series.

The problem is as follows: There are times when a limiting value of a quotient is required, but the numerator and denominator both tend to zero in the limit.

$$\lim_{x \to a} \frac{f(x)}{g(x)}, \quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

This is a so-called *indeterminate form* (although the term is rather misleading):

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim \frac{0}{0}$$

L'Hopital's Rule starts from expanding f and g in Taylor series about the point $x = a$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(\frac{f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots}{g(a) + (x-a)g'(a) + \frac{(x-a)^2}{2!} g''(a) + \dots} \right)$$

By definition $f(a), g(a) \to 0$ as $x \to a$. Then divide through by $(x - a)$, and all terms will tend to zero as $x \to a$, EXCEPT: $f'(a)$ and $g'(a)$.

So, L'Hopital's rule is: if the numerator and the denominator of a quotient separately tend to zero, then the limit of the quotient is the same and the limit of the quotient of the functions' derivatives.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

If, even after differentiation, numerator and denominator are still both zero, then the process is simply repeated. Sometimes it is necessary to differentiate several times to avoid the indeterminate form.

Example

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim\frac{0}{0}$$

Applying L'Hopital's Rule:

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left(\frac{\cos x}{1}\right) = 1$$

Example

$$\lim_{x\to 0}\left(\frac{x-\sin x}{x^2}\right) = \lim\frac{0}{0}$$

Applying L'Hopital's rule:

$$\lim_{x\to 0}\left(\frac{x-\sin x}{x^2}\right) = \lim_{x\to 0}\left(\frac{1-\cos x}{2x}\right) = \lim\frac{0}{0}$$

So, applying L'Hopital's rule *again*:

$$\lim_{x\to 0}\left(\frac{x-\sin x}{x^2}\right) = \lim_{x\to 0}\left(\frac{1-\cos x}{2x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{2}\right) = 0$$

Example

L'Hopital's Rule may also be applied to an indeterminate form such as $\infty - \infty$, as in the following:

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) = \lim_{x\to 0}\left(\frac{1-\cos x}{x\cos x+\sin x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{2\cos x - x\sin x}\right) = 0$$