SUMMARY SHEET: INTEGRATION
Engineering Maths II (MAE111), 2011

Integration is the inverse operation to differentiation:

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\,dx\right) = \int \frac{\mathrm{d}f}{\mathrm{d}x}\,dx$$

Indefinite Integral:

$$I(x) = \int f(x)\,dx$$

The function $f(x)$ is called the *integrand*. Remember to introduce a constant of integration in this case!

Definite Integral:

$$\int_a^b f(x)\,dx = I(b) - I(a)$$

Standard Integrals: Use table of standard integrals. A table of derivatives can be used as well, but remember that integration is like the anti-derivative.

Completing the Square: Useful for putting quadratic expressions into a standard form before integration.
Add the square of half the coefficient of x, and subtract the same.

$$\begin{aligned} x^2 + 6x + 1 &= x^2 + 6x + \left(\frac{6}{2}\right)^2 + 1 - \left(\frac{6}{2}\right)^2 \\ &= (x^2 + 6x + 9) - 8 \\ &= (x+3)^2 - \left(\sqrt{8}\right)^2 \end{aligned}$$

Substitution: When integrand if $f(z(x))$, then try writing

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}z}\,\mathrm{d}z$$

and substituting so that integral is over z, instead of x. Substitution will also work when integrand is a product of the of the form

$$f'(x)\ f(x)$$

or a quotient of the form

$$\frac{f'(x)}{f(x)}$$

Integration by Parts:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

Split integral by choosing u and $\mathrm{d}v$. Make sure that the choice makes the integral easier! Example: By choosing $u = \ln x$, so $\mathrm{d}u = (1/x)\mathrm{d}x$, and $\mathrm{d}v = x^2\mathrm{d}x$, so that $v = (x^3/3)$:

$$I = \int x^2 \ln x\,\mathrm{d}x = \ln x\,\frac{x^3}{3} - \frac{1}{3}\int x^3\frac{1}{x}\mathrm{d}x = \ldots$$

Partial Fractions: When integrand is quotient such as $f(x)/g(x)$, then split using partial fractions. Factorize denominator, then linear factors of form $(ax+b)$ produce terms of form $\frac{A}{(ax+b)}$, repeated factors as $(ax+b)^2$ produce terms as $\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2}$, quadratic factors such as (ax^2+bx+c) produce terms like $\frac{Ax+B}{ax^2+bx+c}$. Introduce PF terms, then equate to original expression to find values for A, B, etc.

Integration of some Trig Functions: For integrands involving powers of $\sin$ or $\cos$, then use following rule: For even powers use double angle formulae

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

(maybe several times). For odd powers split one power off, and use $\cos^2 x + \sin^2 x = 1$ on the rest. Example:

$$\int \sin^3 x\,\mathrm{d}x = \int \sin^2\ \sin x\mathrm{d}x = \int (1-\cos^2 x)\sin x\,\mathrm{d}x = \ldots$$

Useful Trig substitution: Substituting $t = \tan x$, leads to $\sin x = \frac{t}{\sqrt{1+t^2}}$, and $\cos x = \frac{1}{\sqrt{1+t^2}}$, and $\mathrm{d}x = \frac{\mathrm{d}t}{1+t^2}$.

Another Trig substitution: Substituting $t = \tan(x/2)$, leads to $\sin(x/2) = \frac{t}{\sqrt{1+t^2}}$, and $\cos(x/2) = \frac{1}{\sqrt{1+t^2}}$, and $\mathrm{d}x = \frac{2\mathrm{d}t}{(1+t^2)}$. By double angle formulae one obtains $\sin x = \frac{2t}{1+t^2}$, and $\cos x = \frac{1-t^2}{1+t^2}$.

Text Book References

	Mathematics for Engineers Croft & Davison	Engineering Mathematics Stroud (4th & 5th Editions)
Integration	Ch 13	Programme 15, 16

Table of Standard Integrals

$f(x)$	$\int f(x)dx$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}, x > 0$	$\ln\|x\|$
e^{ax}	$\frac{1}{a}e^{ax}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln\|\cos x\| = \ln\|\sec x\|$
$\cot x$	$\ln\|\sin x\|$
$\sec x$	$\ln\|\sec x + \tan x\| = \ln\tan\|(\frac{\pi}{4} + \frac{1}{2}x)\|$
$\operatorname{cosec} x$	$-\ln\|\operatorname{cosec} x + \cot x\| = \ln\|\tan(\frac{1}{2}x)\|$
$\sec^2 x$	$\tan x$
$\operatorname{cosec}^2 x$	$-\cot x$
$\sec x \tan x$	$\sec x$
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\operatorname{sech}^2 x$	$\tanh x$
$\operatorname{cosech}^2 x$	$-\coth x$
$\tanh x$	$\ln\cosh x$
$\coth x$	$\ln\|\sinh x\|$
$\operatorname{sech} x \tanh x$	$-\operatorname{sech} x$
$\operatorname{cosech} x \coth x$	$-\operatorname{cosech} x$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\frac{a+x}{a-x}, (x < a)$
$\frac{1}{\sqrt{(a^2-x^2)}}$	$\sin^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{(x^2+a^2)}}$	$\sinh^{-1}\left(\frac{x}{a}\right) = \ln\left[\frac{x+\sqrt{(x^2+a^2)}}{a}\right]$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right) = \ln\left[\frac{x+\sqrt{(x^2-a^2)}}{a}\right]$
$\frac{1}{x\sqrt{(x^2-a^2)}}$	$\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$