MAE111 Engineering Mathematics II (2003/2004 Sem. 1)

SECTION 5: INTEGRATION

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Section 5

Integration

5.1 Concepts and Definitions

Figure of area under $f(x)$

Figure of area under $f(x)$ with area strips

For a curve $y = f(x)$ there is a definite concept of the **area** under the curve between $x = a$ and $x = b$, i.e., the area bounded by the straight lines $y = 0$, $x = a$, and $x = b$, and the curve $y = f(x)$.

There is no simple formula for this area (as for, e.g., a circle), but an approximation to the area can be found by splitting the interval (a, b) into many vertical strips, at positions x_i and with widths Δx_i (where i labels each strip with an integer). Within a strip i the function $f(x)$ takes a range of values, and one value is chosen and becomes the quantity y_i. (The choice is arbitrary, it just needs to be in the right range.) The area under the curve is then approximately the sum of the areas of all the rectangular strips.

$$A \approx \sum_i y_i \Delta x_i$$

As the number of strips increases and the Δx_i become smaller, then the approximation gets better and better. At the same time, the quantities y_i approach the value of the function at x_i (since the range of $f(x)$ in each strip decreases as the strip narrows). The limiting value of this process is called the **definite integral** of the function $f(x)$ over the range $x = a$ to $x = b$:

$$A = \int_a^b f(x)\,\mathrm{d}x$$

The quantities a and b are known as the lower and upper limits of the integral, respectively. By convention if $f(x) < 0$ then it makes a negative contribution to the area.

Now consider what happens if the lower limit is fixed, but the upper limit is allowed to vary. The area is now a function of the position of the upper limit, and so we denote that upper limit using the variable x.

$$I(x) = \int_a^x f(u)\,\mathrm{d}u$$

The variable for the integration has been changed to u, so as not to get confused with the x of the upper limit. In fact the integration variable can have any name, since it does not appear in the result. (It is sometimes called a dummy variable.)

If the lower limit is $\tilde{a}$ instead of a then we can write

$$\tilde{I}(x) = \int_{\tilde{a}}^{x} f(u)\ \mathrm{d}u = \int_{\tilde{a}}^{a} f(u)\ \mathrm{d}u + \int_{a}^{x} f(u)\ \mathrm{d}u = C + I(x)$$

Where the constant C is just a number, since it is the result of a definite integral. This shows that any two such integrals only differ from each other only by some constant value, which is usually called the *constant of integration.* Thus we define $I(x)$, which is called the **indefinite integral** of the function $f(x)$, as

$$I(x) = \int f(x)\ \mathrm{d}x$$

The indefinite integral can only be evaluated to within some arbitrary constant value.

It follows that

$$\int_{a}^{b} f(x)\ \mathrm{d}x = I(b) - I(a)$$

Next, consider how $I(x)$ varies as x increases. Let $x \to x + \delta x$, then

$$I(x + \delta x) = I(x) + f(\tilde{x})\delta x$$

where $f(\tilde{x})\delta x$ is the area of the rectangular stript to be added, and $\tilde{x}$ is some position within the strip. Now, what happens as the strip which is being added gets narrower and narrower? As $\delta x \to 0$, then $\tilde{x} \to x$ and, rearrranging, we have

$$f(x) = \lim_{\delta x \to 0} \left(\frac{I(x + \delta x) - I(x)}{\delta x} \right)$$

The rhs is none other than the definition of the derivative of the indefinite integral!

Therefore, we have found that the derivative of the indefinite integral of the function $f(x)$ is (again) the function $f(x)$.

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \left(\int f(x)\ \mathrm{d}x \right) = \int \frac{\mathrm{d}f}{\mathrm{d}x}\ \mathrm{d}x$$

In other words: *Differentiation is the inverse operation to integration.* Or, in even more other words: integration is like taking the *anti-derivative*.

5.2 Use of Standard Integrals

Once the derivative of a function is known, then there is corresponding integral which is known. So a table of derivatives can also be used (in reverse) as a table of differentials.

For example:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{k+1} + C) = (k+1)x^k \quad \Rightarrow \int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$$

Note that this rule only applies for $n \neq -1$.

5.2.1 Standard Forms Involving Squares

A number of standard forms have terms which are like the sum or difference of squares. For example:

$$I = \int \frac{\mathrm{d}Z}{Z^2 + A^2} = \frac{1}{A} \tan^{-1} \left(\frac{Z}{A} \right) + C$$

Manipulating Integrand to Use a Standard Form

A useful technique is **completing the square.**

Consider

$$x^2 + 4x + 2$$

We want to rewrite this as a squared quantity and a remainder.

The rule is:
Add the square of half the coefficient of x, and subtract the same.

$$\begin{aligned} x^2 + 4x + 2 &= x^2 + 4x + \left(\frac{4}{2}\right)^2 + 2 - \left(\frac{2^2}{2}\right) \\ &= x^2 + 4x + 4 - 2 \\ &= (x+2)^2 - 2 \end{aligned}$$

Now the expression is in the form $Z^2 - A^2$, which can be used in some of the standard form integrals. (The change of variable from x to Z is straightforward, since it is a linear relationship.)

Example

$$I = \int \frac{1}{x^2 + 10x + 30}\,\mathrm{d}x$$

Completing the square

$$\begin{aligned} x^2 + 10x + 30 &= x^2 + 10x + \left(\frac{10}{2}\right)^2 + 30 - \left(\frac{10}{2}\right)^2 \\ &= (x+5)^2 + \left(\sqrt{5}\right)^2 \end{aligned}$$

Rewriting the integral allows us to use the standard form immediately:

$$I = \int \frac{1}{(x+5)^2 + \left(\sqrt{5}\right)^2}\,\mathrm{d}x = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x+5}{\sqrt{5}}\right) + C$$

5.3 Function of a Function of x

When the integrand is of the form

$$f(z(x))$$

then try writing

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}z}\mathrm{d}z$$

and substitute so that the integral is over z.

Example

This kind of substitution will always work when $z(x)$ is a *linear* function of x.

$$I = \int (5x+3)^3 \mathrm{d}x \quad z = 5x + 3 \quad \mathrm{d}z = \frac{\mathrm{d}z}{\mathrm{d}x}\mathrm{d}x = 5\mathrm{d}x$$

So,

$$I = \int z^3 \frac{1}{5}\mathrm{d}z = \frac{1}{5}\frac{z^4}{4} + C = \frac{1}{20}(5x+3)^4 + C$$

Example

$$I = \int e^{kx} \mathrm{d}x, \quad z = kx, \quad \mathrm{d}z = k\mathrm{d}x$$

So,

$$I = \int \frac{e^z}{k} \mathrm{d}z = \frac{1}{k} e^z + C = \frac{1}{k} e^{kx} + C$$

Example

$$I = \int x \sin(x^2) \mathrm{d}x, \quad z = x^2, \quad \mathrm{d}z = 2x\mathrm{d}x$$

So,

$$I = \int x \sin(z) \frac{\mathrm{d}z}{2x} = \frac{-1}{2} \cos(z) + C = \frac{-\cos(x^2)}{2} + C$$

Example

The same kind of substitution also works in some other circumstances.

$$I = \int \frac{2x+3}{x^2+3x-5} \mathrm{d}x, \quad z = x^2 + 3x - 5, \quad \mathrm{d}z = (2x+3)\mathrm{d}x$$

Note, that in the quotient integrand the numerator is the derivative of the denominator.
So,

$$I = \int \frac{\mathrm{d}z}{z} = \ln z + C = \ln(x^2 + 3x - 5) + C = A \ln(x^2 + 3x - 5)$$

Note how an additive constant of integration can appear as a *multiplicative* factor on a logarithmic function.

Example

$$I = \int \sin x \ \cos x \ \mathrm{d}x, \quad z = \sin x, \quad \mathrm{d}z = \cos x \ \mathrm{d}x$$

In this case, note that in the product integrand, one part is the derivative of the other.
So,

$$I = \int z \, \mathrm{d}z = \frac{1}{2} z^2 + C = \frac{1}{2} \sin^2 x + C$$

In cases such as this one can use a different notation, so the substitution is not explicit:

$$I = \int \sin x \ \cos x \ \mathrm{d}x, \quad d(\sin x) = \cos x \ \mathrm{d}x$$

$$I = \int \sin x \ d(\sin x) = \frac{1}{2} \sin^2 x + C$$

Summary

For an integrand which is function of a function of x, then try substitution (i.e., change of variable). Substitution also works in the following cases:
If the integrand is a product, then check if it is in the form:

$$f'(x).f(x)$$

If the integrand is a quotient, then check if it is in the form:

$$\frac{f'(x)}{f(x)}$$

5.4 Integration by Parts

Integration by parts is the equivalent of the Product Rule for differentiation. Consider the following, for $u(x)$ and $v(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Now integrate both sides with respect to x:

$$\int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\,\mathrm{d}x = uv = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

And rearrange:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

This is sometimes written:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

which is easier to remember.

The left hand side integrand is a product of two terms (u and the derivative of v), whilst the right hand side contains another integral whose integrand is also a product. Now, clearly, using this technique is only useful if the integral on the rhs is easier than the integral on the lhs! There is also the need to find v from its derivative, so we want that to be easy as well! So, when using this technique, the choice of the functions u and dv are crucial.

Example

Consider,

$$I = \int x^2 \ln x\,\mathrm{d}x$$

We make the following choice:

$$u = \ln x, \quad \Rightarrow \mathrm{d}u = \frac{1}{x}$$

$$x^2\,\mathrm{d}x = \mathrm{d}v, \quad \Rightarrow v = \frac{x^3}{3}$$

Now, integrating by parts:

$$I = \int x^2 \ln x\,\mathrm{d}x = \ln x\frac{x^3}{3} - \frac{1}{3}\int x^3\frac{1}{x}\mathrm{d}x$$

Now the last integral is much easier.

$$I = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C = \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C$$

In this case we chose $u = \ln x$, since it doesn't appear as a standard integral, but it can be differentiated straightforwardly.

Example

In the last example, we chose not to try to integrate $\ln x$. But, it turns out that we can use integration by parts,

$$I = \int \ln x\,\mathrm{d}x$$

where we choose $u = \ln x$, and $\mathrm{d}v = 1.\,\mathrm{d}x$ (so $v = x$):

$$I = \int \ln x.1\,\mathrm{d}x = x\ln x - \int x\frac{1}{x}\mathrm{d}x$$

Therefore,

$$\int \ln x\,\mathrm{d}x = x\ln x - x + C$$

Example

Here is an example, where integration by parts has to be done twice before reaching the answer.

$$I = \int e^{5x}\sin 3x\,\mathrm{d}x$$

We use:

$$u = e^{5x}, \quad \mathrm{d}v = \sin 3x \quad \Rightarrow v = \frac{-\cos 3x}{3}$$

So,

$$I = e^{5x}\left(\frac{-\cos 3x}{3}\right) + \int \frac{\cos 3x}{3}\,5e^{5x}\,\mathrm{d}x$$

And now integrate by parts again on the remaining integral (with $\mathrm{d}v = \cos 3x$):

$$I = \frac{-1}{3}e^{5x}\cos 3x + \frac{5}{3}\left[e^{5x}\frac{\sin 3x}{3} - \frac{5}{3}\int \sin 3x\, e^{5x}\,\mathrm{d}x\right]$$

Note that now the remaining integral is none other than the original one! So, we can write

$$I = \frac{-1}{3}e^{5x}\cos 3x + \frac{5}{9}e^{5x}\sin 3x - \frac{25}{9}I$$

and rearranging:

$$\frac{34}{9}I = \frac{e^{5x}}{3}\left[\frac{5}{3}e^{5x}\sin 3x - \cos 3x\right] + C'$$

So, finally:

$$I = \frac{3e^{5x}}{34}\left[\frac{5}{3}e^{5x}\sin 3x - \cos 3x\right] + C$$

Example

Here is a *bad* example! If you choose badly for the functions u and v then integration by parts only makes things worst. . .

Consider:

$$I = \int x^3 e^x\,\mathrm{d}x$$

and choose $\mathrm{d}v = x^3$ and $u = e^x$, so

$$I = \frac{x^4}{4} - \int \frac{x^4}{4}e^x\,\mathrm{d}x$$

This is true, but doesn't get us any nearer the answer!

5.5 Integration Using Partial Franctions

If the integrand is a *rational function* (i.e, a quotient of two polynomial functions) then try the following

- factorize the denominator
- rewrite as a sum of terms (ie using partial fractions)
- integrate the individual terms (should be easier!)

So the integrand is of the form:

$$\frac{f(x)}{g(x)}\left(=\frac{\text{Numerator}}{\text{Denominator}}\right)$$

The rules of partial fractions are:

- *Numerator* must be of lower degree than *denominator* – if not, then divide out (see Stroud F8.20).
- Factorize the denominator
- Linear factors of form $(ax+b)$ produce a partial fraction of form

$$\frac{A}{ax+b}$$

- Repeated factors as $(ax+b)^2$ produce terms like

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

- Quadratic factors such as (ax^2+bx+c) produce a term like

$$\frac{Ax+B}{ax^2+bx+c}$$

In the above, A and B are constants that are found by putting on a common denominator, and equating coefficients of the powers of x.

Example

$$I=\int\frac{x+1}{x^2-3x+2}\,\mathrm{d}x$$

Rewriting the quotient using partial fractions:

$$\frac{x+1}{x^2-3x+2}=\frac{x+1}{(x-2)(x-1)}=\frac{A}{x-1}+\frac{B}{x-2}$$

A and B are found by multiplying up and equating lhs and rhs denominators.

$$A(x-2)+B(x-1)=x+1$$

From the coefficients of powers of x:

$$\begin{aligned} A+B &= 1 \\ 2A+B &= -1 \end{aligned}$$

which has the solution

$$A=-2 \quad B=3$$

(An alternative method is to set $x=1$ to get rid of B term, to find $A=-2$, then set $x=2$ to get rid of A term to find $B=3$.)

The integral then becomes

$$\begin{aligned} I &= \int \frac{3}{x-2}\mathrm{d}x - \int \frac{2}{x-1}\mathrm{d}x \\ I &= 3\ln(x-2) - 2\ln(x-1) + C \end{aligned}$$

Example

$$I = \int \frac{x^2+3x+3}{(x^2+4x+4)(x+1)}\,\mathrm{d}x$$

$$\frac{x^2+3x+3}{(x^2+4x+4)(x+1)} = \frac{(Ax+B)}{(x^2+4x+4)} + \frac{C}{x+1}$$

So, after multiplying up and equating denominators:

$$x^2+3x+3 = (Ax+B)(x+1) + C(x^2+4x+4)$$

To get rid of first term on rhs, set $x=-1$, then we are left with

$$1-3+3 = C(1-4+4) \quad \Rightarrow C = 1$$

From coefficients of x^2:

$$1 = A + C \quad \Rightarrow A = 0$$

And from coefficients of 1 (i.e. x^0)

$$3 = B + 4C \quad \Rightarrow B = -1$$

So our integral can be rewritten as

$$I = \int \frac{(-1)}{(x^2+4x+4)}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{x+1}$$

Now, $x^2+4x+4=(x+2)^2$, so

$$\begin{aligned} I &= \int \frac{(-1)}{(x+2)^2}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{x+1} \\ I &= \frac{1}{(x+2)} + \ln(x+1) + K \end{aligned}$$

(Using K for the constant of integration, so as not to get confused with the partial fractions constants.)

5.6 Integration of Some Trigonometric Functions

How can one integrate functions which are powers of either $\sin$ or $\cos$?

For

$$\int \sin^2 x\,\mathrm{d}x, \quad \text{and} \quad \int \cos^2 x\,\mathrm{d}x$$

one uses the double angle formulae;

$$\cos 2x = 1 - 2\sin^2 x \quad \Rightarrow \sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\cos 2x = 2\cos^2 x - 1 \quad \Rightarrow \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

For example,

$$\int \sin^2 x \, \mathrm{d}x = \int \frac{1}{2}(1-\cos 2x)\mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

For

$$\int \sin^n x \, \mathrm{d}x, \quad \text{and} \quad \int \cos^n x \, \mathrm{d}x$$

where n is an integer, the rule is:

- For <u>even</u> powers use the double angle formulae (maybe several times).
- For <u>odd</u> powers split one power off, and use $\cos^2 x + \sin^2 x = 1$.

Example

$$\int \sin^3 x \, \mathrm{d}x = \int \sin^2 x \sin x \, \mathrm{d}x = \int (1-\cos^2 x) \sin x \, \mathrm{d}x$$

$$\int \sin^3 x \, \mathrm{d}x = \int \sin x \, \mathrm{d}x + \int \cos^2 x \, d(\cos x)$$

The last step follows from the fact that $d(\cos x) = -\sin x \, \mathrm{d}x$. And finally:

$$\int \sin^3 x \, \mathrm{d}x = -\cos x + \frac{\cos^3 x}{3} + C$$

5.7 Useful Trigonometric Substitutions

There are a number of types of integrals, involving trigonometric functions that can be solved with substitutions which rely on the properties of a right-handed triangle with $\tan x = t$.

Figure of triangle, sides 1, t, and $\sqrt{1+t^2}$.

$$\begin{aligned} \tan x &= t \\ \sin x &= \frac{t}{\sqrt{1+t^2}} \\ \cos x &= \frac{1}{\sqrt{1+t^2}} \end{aligned}$$

It follows that

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \sec^2 x = 1 + \tan^2 x = 1 + t^2 \quad \Rightarrow \mathrm{d}x = \frac{\mathrm{d}t}{1+t^2}$$

Example

$$I = \int \frac{1}{3+\cos^2 x} \, \mathrm{d}x$$

Using the $t = \tan x$ substitution:

$$3 + \cos^2 x = 3 + \frac{1}{1+t^2} = \frac{3t^2+4}{1+t^2}$$

So,

$$\begin{aligned} I &= \int \frac{(1+t^2)}{(3t^3+4)} \frac{\mathrm{d}t}{(1+t^2)} \\ &= \frac{1}{3} \int \frac{\mathrm{d}t}{(t^2+\frac{4}{3})} \\ &= \frac{1}{3} \frac{\sqrt{3}}{2} \tan^{-1}\left(\frac{t}{2/\sqrt{3}}\right) + C \\ &= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}\tan x}{2}\right) + C \end{aligned}$$

A similar substitution can be used for integrals of the form:

$$\int \frac{\mathrm{d}x}{a + b\sin x + c\cos x}$$

where a, b, and c are constants. Using:

$$\begin{aligned} \tan\left(\frac{x}{2}\right) &= t \\ \sin\left(\frac{x}{2}\right) &= \frac{t}{\sqrt{1+t^2}} \\ \cos\left(\frac{x}{2}\right) &= \frac{1}{\sqrt{1+t^2}} \end{aligned}$$

And by double angle formulae:

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2t}{1+t^2}$$
$$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \frac{(1-t^2)}{(1+t^2)}$$

And so on (see Stroud p646) ...

5.8 Proof of Some Standard Forms

There are a number of standard integrals, and this section gives proofs of some of the more unfamiliar.

NOTE: This section will only be briefly covered in the lectures. However, the material in this section should be read through, since it contains examples of integration techniques.

It is also important to realise that if the integrand can be manipulated into to the right form, then the standard integrals can be applied to an even wider set of problems.

5.8.1 Standard Form: $\int (Z^2 + A^2)^{-1}\,\mathrm{d}Z$

Use the substitution

$$Z = A\tan\theta \quad \Rightarrow \mathrm{d}Z = A\sec^2\theta\,\mathrm{d}\theta$$

Also

$$Z^2 + A^2 = A^2(1+\tan^2\theta) = A^2\sec^2\theta$$

So,

$$I = \int \frac{\mathrm{d}Z}{Z^2 + A^2} = \int \frac{A\sec^2\theta}{A^2\sec^2\theta}\,\mathrm{d}\theta = \frac{\theta}{A} + C$$

But...

$$\theta = \tan^{-1}\left(\frac{Z}{A}\right)$$

So

$$I = \int \frac{\mathrm{d}Z}{Z^2 + A^2} = \frac{1}{A}\tan^{-1}\left(\frac{Z}{A}\right) + C$$

Example

(See example earlier.)

5.8.2 Other Standard Form by Substitution

Full details are in the text book.

Substitution $Z = A\sin\theta$

$$\int \frac{\mathrm{d}Z}{\sqrt{A^2 - Z^2}} = \sin^{-1}\left(\frac{Z}{A}\right) + C$$

$$\int \sqrt{A^2 - Z^2}\,\mathrm{d}Z = \frac{A^2}{2}\left\{\sin^{-1}\left(\frac{Z}{A}\right) + \frac{Z\sqrt{A^2 - Z^2}}{A^2}\right\} + C$$

Substitution $Z = A\sinh\theta$

$$\int \frac{\mathrm{d}Z}{\sqrt{Z^2 + A^2}} = \sinh^{-1}\left(\frac{Z}{A}\right) + C$$

$$\int \sqrt{Z^2 + A^2}\,\mathrm{d}Z = \frac{A^2}{2}\left\{\sinh^{-1}\left(\frac{Z}{A}\right) + \frac{Z\sqrt{Z^2 + A^2}}{A^2}\right\} + C$$

Substitution $Z = A\cosh\theta$

$$\int \frac{\mathrm{d}Z}{\sqrt{Z^2 - A^2}} = \cosh^{-1}\left(\frac{Z}{A}\right) + C$$

$$\int \sqrt{Z^2 - A^2}\,\mathrm{d}Z = \frac{A^2}{2}\left\{\frac{Z\sqrt{Z^2 - A^2}}{A^2} - \cosh^{-1}\left(\frac{Z}{A}\right)\right\} + C$$

5.8.3 Other Standard Forms by Partial Fractions

Consider

$$I = \int \frac{\mathrm{d}Z}{Z^2 - A^2}$$

Now,

$$\frac{1}{Z^2 - A^2} = \frac{1}{(Z - A)(Z + A)} = \frac{P}{Z - A} + \frac{Q}{Z + A}$$

Using partial fractions:

$$P(Z+A)+Q(Z-A)=1 \quad \Rightarrow P=\frac{1}{2A}, \quad Q=\frac{-1}{2A}$$

Therefore

$$\begin{aligned} I &= \frac{1}{2A}\int \frac{\mathrm{d}Z}{(Z-A)} - \frac{1}{2A}\int \frac{\mathrm{d}Z}{(Z+A)} \\ &= \frac{1}{2A}\ln(Z-A) - \frac{1}{2A}\ln(Z+A) + C \\ &= \frac{1}{2A}\ln\left(\frac{Z-A}{Z+A}\right) + C \end{aligned}$$

Similarly, one can show that

$$\int \frac{\mathrm{d}Z}{A^2-Z^2} = \frac{1}{2A}\ln\left(\frac{A+Z}{A-Z}\right)+C$$

Example

$$I = \int \frac{\mathrm{d}z}{3-x^2}$$

So, to use the standard form must have $A^2=3$, or $A=\sqrt{3}$:

$$I = \frac{1}{2\sqrt{3}}\ln\left(\frac{\sqrt{3}+x}{\sqrt{3}-x}\right)+C$$

Example

(Need to take care of signs!)

$$I = \int \frac{1}{3+6x-x^2}\mathrm{d}x$$

Completing the square

$$\begin{aligned} 3+6x-x^2 &= 3-(x^2-6x) \\ &= 3-\left(x^2-6x+\left(\frac{6}{2}\right)^2\right)+\left(\frac{6}{2}\right)^2 \\ &= 3-(x-3)^2+9 \\ &= 12-(x-3)^2 \end{aligned}$$

So, for standard form $A=\sqrt{12}=2\sqrt{3}$, and it follows that

$$I = \frac{1}{4\sqrt{3}}\ln\left(\frac{2\sqrt{3}+x-3}{2\sqrt{3}-x+3}\right)+C$$

5.9 Reduction Formulae

5.9.1 Introduction: Recursion

A common concept in mathematics (and computer programming) is the idea of using an operation repeatedly on itself, until the desired result is obtained. This is called recursion.

For example, consider the usual way of defining the factorial function

$$n! = (1)(2)(3)\dots(n-2)(n-1)n$$

In other words, just list out all the integers from 1 to n and multiply them all together.

But, the following is an equivalent *recursive* definition

$$n! = n(n-1)!, \quad 1! = 1.$$

In this case the definition of $n!$ is in terms of $(n-1)!$. But what is the definition for $(n-1)!$? Well, that can be defined in terms of $(n-2)!$ using the same of definition just by replacing n by $(n-1)$. And so on, until we reach the terminating part of the definition $1! = 1$.

So, a recursive definition often contains a term labelled by n, which is defined in terms labelled with $(n-1)$ (and/or maybe other terms labelled less than n). There also has to be a terminal (or final) part of the definition which stops the recursion carrying on forever. Recursive definitions are used by repeatedly applying the recursive part, until the terminal part can be used, and that completes the process.

5.9.2 Integration by Use of Reduction Formulae

In the last section we have seen how the integral of $\sin^3 x$ can be reduced to an answer with a term which is an integral of $\sin x$. In other words, the power that appears in the original integrand has been reduced. This points us to a generalization, where we examine integrands which have an *arbitrary* power.

For example, consider the integral

$$I_n = \int x^n e^x \, \mathrm{d}x$$

Note that we label the integral with n to denote that the integrand contains x^n. Integrating by parts with $u = x^n$ and $\mathrm{d}v = e^x \mathrm{d}x$:

$$I_n = x^n e^x - \int n x^{n-1} e^x \, \mathrm{d}x = x^n e^x - n \int x^{n-1} e^x \, \mathrm{d}x$$

Now, the integral on the rhs is basically the same as I_n, but with $(n-1)$ instead of n, so we can see that it can be written as just I_{n-1}. We then find the following:

$$I_n = x^n e^x - I_{n-1}$$

This is known as a **reduction formula.** It is a relationship between integrals of the same form, but enables an integral containing the power n to be written in terms of one containing $(n-1)$. Sometimes, the reduction formula might be a relationship between more than two integrals, i.e, I_n in terms of I_{n-1} and I_{n-2}.

In itself this is just an interesting relationship, but it becomes a powerful technique once one notes that I_0 is a simple integral that can be found immediately:

$$I_0 = \int x^0 e^x \, \mathrm{d}x = \int e^x \, \mathrm{d}x = e^x + C$$

Since I_0 is known, an application of the reduction formula means that we can easily find I_1:

$$I_1 = x^1 e^x - I_0 = x e^x - e^x + C$$

And repeating the procedure means we can find I_2:

$$I_2 = \int x^2 e^x \, \mathrm{d}x = x^2 e^x - I_1 = x^2 e^x - x e^x + e^x + C$$

The whole beauty of this technique is that we can find ever more complicated integrals without doing any more integration!

Example

Consider

$$I_n = \int x^n \cos x \, \mathrm{d}x$$

By parts, with $u = x^n$ and $\mathrm{d}v = \cos x \mathrm{d}x$:

$$I_n = x^n \sin x - \int n x^{n-1} \sin x \, \mathrm{d}x$$

But the rhs does not contain a term of the right form, so we integrate by parts a second time

$$\begin{aligned} I_n &= x^n \sin x + n x^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \, \mathrm{d}x \\ I_n &= x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2} \end{aligned}$$

This is now in the form of a reduction formula. Notice, that because the reduction formula gives I_n in terms of I_{n-2}, the odd and even terms start from different integrals (I_1 and I_0, respectively) and both can be easily found.

If the integral to be evaluated has limits, then the corresponding reduction formula can be used (i.e., apply the limits to the reduction formula directly). This is often simpler than keeping the general form and then substituting later. For example consider:

$$I_n = \int_0^\pi x^n \cos x \, \mathrm{d}x$$

Then applying limits to the general reduction formula:

$$\begin{aligned} I_n &= \left[x^n \sin x + n x^{n-1} \cos x \right]_0^\pi - n(n-1) I_{n-2} \\ I_n &= \left[0 + n \pi^{n-1}(-1) - (0+0) \right] - n(n-1) I_{n-2} \\ I_n &= -n \pi^{n-1} - n(n-1) I_{n-2} \end{aligned}$$