MAE111 Engineering Mathematics II
(2003/2004 Sem. 1)

SECTION 6: APPLICATIONS OF INTEGRATION

D. Burgess
School of Mathematical Sciences
Queen Mary, University of London,
Mile End Road, London E1 4NS, U.K.
`D.Burgess@qmul.ac.uk`

If you find a mistake, then please tell the author!

November 25, 2003

Section 6

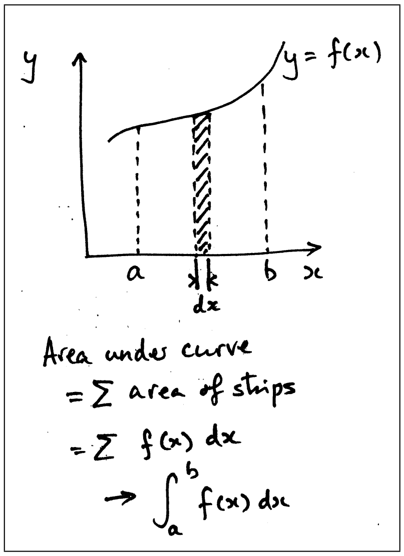
Applications of Integration

6.1 Double Integrals

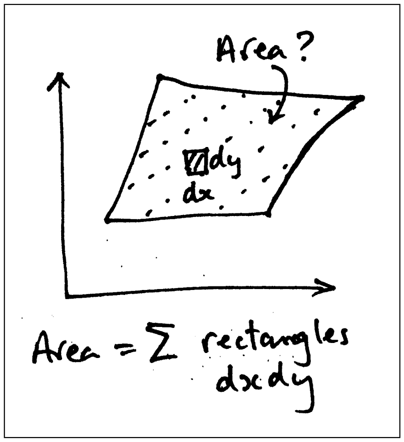
The notion of an integral as the area under the curve has been introduced by considering that area as approximated by a sum of rectangular strips, width Δx and height $f(x)$:

$$A \approx \sum f(x)\Delta x$$

As the width of the strips tend to zero, and their number increases, then the result of this limiting process was defined as the integral of the function.

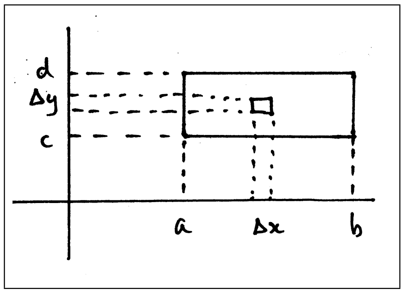


Now, suppose that instead of summing rectangular strips (parallel to the y axis) to find the area under a curve, we consider how we would find the area bounded by some closed curve. The point here is that we want to generalize from the case where the area is simply bounded by the x and y axes.



As a simple case, consider the (rectangular) area bounded by the four lines $x = a$, $x = b$, $y = c$, and $y = d$. One way to approximate this area, is to imagine the area divided into small rectangles of sides Δx and Δy at position (x, y). Then, the area would be the sum of such rectangles in the y direction (at fixed x), and then the sum of such quantities over the range of x.

$$A \approx \sum_x \left(\sum_y \Delta y \right) \Delta x$$



As before, letting the number of such small rectangles increase, and their size decrease, one approaches a limit, but now, since there are two summations, the result is a double integral

$$A = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \mathrm{d}y \right) \mathrm{d}x$$

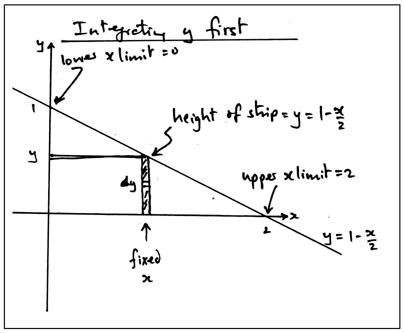
Note that the inner integral is at a *fixed* value of x. Also the brackets around the inner integral are not normally shown. Carrying out the integral over y:

$$A = \int_{x=a}^{x=b} (d-c)dx = (b-a)(d-c)$$

For this rectangular region the answer by integration is obviously correct!

Double integration is more problematic when the area to be integrated over, has limits which depend on x and y. Here it becomes crucial to consider the *order of integration*, i.e., whether the x or y integration is carried out first, and the consequent effect on the limits of integration.

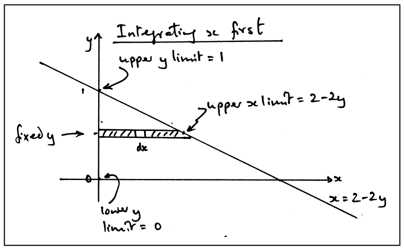
Consider the area bounded by $x = 0$, $y = 0$, and $y = 1 - \frac{x}{2}$. Using double integrals, if the y integration is done first, then it is done at fixed x, and so the limit must be in terms of x (not y). Conversely, if the x integration is done first, then it is done at fixed y, and so the limit must be in terms of y (not x). The limits for the outer integration must always just be numbers, since in the end the result must contain no dependence on either x or y.



$$A = \int_{x=0}^{2} \left(\int_{y=0}^{1-x/2} \mathrm{d}y \right) \mathrm{d}x = \int_{x=0}^{2} \left(1 - \frac{x}{2}\right) \mathrm{d}x = \left[x - \frac{x^2}{4} \right]_0^2 = 1$$

Alternatively, noting that the upper line can be written as $x = 2 - 2y$:

$$A = \int_{y=0}^{1} \left(\int_{x=0}^{2-2y} \mathrm{d}x \right) \mathrm{d}y = \int_{y=0}^{1} (2 - 2y) \ \mathrm{d}y = [2y - y]_0^1 = 1$$



When changing the order of integration, one needs to look carefully at the geometry of the region, and the limits of integration.

One can generalize the concept of double integration to the integration of any function which might be a function of position in the $x - y$ plane. For example: If ρ is the mass per unit area (and it might not be constant in position), then the total mass is the sum of all infinitesimal areas multiplied by ρ at the position (x, y).

$$M = \int \int \rho \, \mathrm{d}y \, \mathrm{d}x$$

As a reminder: the inner integral over y is done at fixed x, in other words x is treated as a constant for the integral over y.

6.1.1 Notation

There are two different kinds of notation to be aware of:

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \mathrm{d}y \ \mathrm{d}x$$

In this case the y integral is done first (ie work from inner integral outwards). (This notation is easier to understand.)

$$\int_{x_1}^{x_2} \mathrm{d}x \int_{y_1}^{y_2} \mathrm{d}y$$

In this case the y integral is done first (ie work from right to left).

Example

Consider a plane surface of a material whose mass per unit area is given by $\rho = x^4$ (eg in units of kg/m^2). Find the total mass of the closed region bounded by the curves $x^2 + y = 8$ (a parabola) and $y = 2x$ (a straight line).

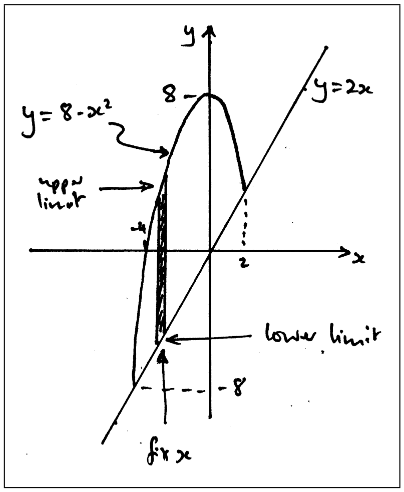
It is vital to sketch this region, to determine the limits of integration. The points of intersection of the straight line and parabola can be found by solving the two equations simultaneously. Substituting $y = 2x$ into $x^2 + y = 8$ gives

$$x^2 + 2x = 8 \quad \Rightarrow x^2 + 2x - 8 = 0 \quad \Rightarrow (x+4)(x-2) = 0$$

So, the intersection points are $(-4, -8)$ and $(2, 4)$. Thus the range of x is from -4 to $+2$.

For any fixed value of x, the range of y is from the straight line $y = 2x$, to the parabola $y = 8 - x^2$, since the straight line is always below the parabola for the region concerned.

Thus we have found: the range of y for fixed x, and the absolute range of x (irrespective of y). This implies that we should do the y integral first, and then the x integral.



Could be replaced by a simpler example, such as:

$$\int_0^2 \int_0^{1-x/2} xy \; dydx.$$

$$M = \int_{x=-4}^{2} \left(\int_{y=2x}^{y=8-x^2} x^4 \, \mathrm{d}y \right) \, \mathrm{d}x$$

Note, how there is a x^4 term in the y integral, but this should be treated as if it were a constant, and can be taken outside the y integral. So, the first step is simply to integrate $\int \mathrm{d}y$ and apply the limits.

$$M = \int_{x=-4}^{2} \left(x^4 \, [y]_{2x}^{8-x^2} \right) \, \mathrm{d}x$$

Applying the limits is straightforward, even though they contain x:

$$M = \int_{x=-4}^{2} \left(x^4 \left[(8 - x^2) - 2x \right] \right) \, \mathrm{d}x$$

So, rearranging:

$$M = \int_{x=-4}^{2} \left(8x^4 - 2x^5 - x^6 \right) \, \mathrm{d}x$$

Integrating now with respect to x:

$$M = \left[\frac{8x^5}{5} - \frac{2x^6}{6} - \frac{x^7}{7} \right]_{-4}^{2}$$

And applying limits:

$$M = 674.742.$$

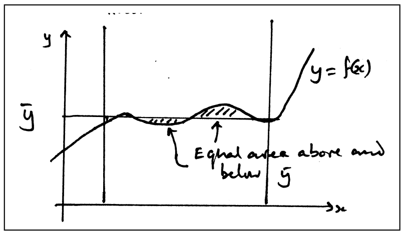
Note, that the x integral could have been done first, but the total integral would have had to be split into two parts, since for $y > 4$ the limits on the x integration would have a different form (ie not depend on the straight line at all).

Usually the order of integration is chosen to make the limits of the region easier, or to make the integral easier.

6.2 Integration Applications

Integration is a process of summing up contributions each of which depends, in some way, on a continuous function. We have seen how ordinary integration can be related to the area under a curve, and how double integration can be related to the area within a closed curve. Now we move on to a number of applications of integration, where the problem involves different kinds of summation, for example, summing volume elements, or line elements.

6.2.1 Mean of Function



If a function represents some quantity such as power consumption as a function of time, then it is useful to define statistical properties of the function. The easiest is the mean. Extending the concept of the mean for a discontinuous quantity, we can define the mean of a function as the integral of the function (i.e., the area under the curve), divided by the interval for the integral:

Mean value $\bar{y}$ of function $y = f(x)$ between $x = a$ and $x = b$

$$\bar{y} = \frac{\int_a^b y \, \mathrm{d}x}{b - a}$$

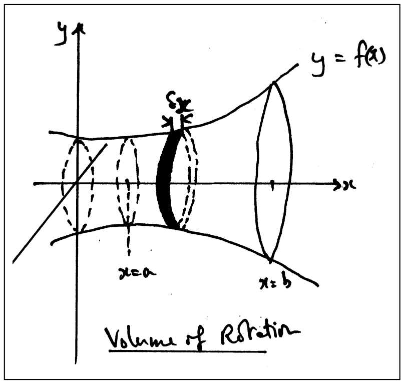
Other statistical quantities, such as the variance, etc, can be expressed as integrals.

6.2.2 Root Mean Square of Function

A useful quantity for a fluctuating quantity is the *root mean square* value. It is a positive quantity, and is frequently used when analysing electrical systems.

Root mean square (RMS) value y_{rms} of function $y = f(x)$ between $x = a$ and $x = b$

$$y_{\mathrm{rms}} = \sqrt{\frac{\int_a^b y^2 \, \mathrm{d}x}{b - a}}$$



6.2.3 Volume of Revolution – About x axis

Consider a curve defined by the function $y = f(x)$. By rotating this curve about the x axis, one generates what is known as a *volume of revolution*. The volume V that is between limits $x = a$ and $x = b$ ($a < b$) can be found by considering that it may be approximated by a sum of the volumes of a set of disks. Each disk at a position x, has a radius $y = f(x)$, and a thickness δx. The volume of each such elemental disk is the circular area of the disk multiplied by its thickness. Summing up all such disks:

$$V \approx \sum_{x=a}^{x=b} \pi y^2 \delta x$$

As the number of disks increases, and their width decreases, then the sum is an ever increasingly better approximation, and in the limit $\delta x \to 0$ the sum is written as the integral

$$V = \int_a^b \pi y^2 \, \mathrm{d}x$$

Example

This example also illustrates how the integration can be done when the curve is given in parametric form.

A curve is given in parametric form: $y = t^2/2$, and $x = 4t - t^2$. What is the volume of revolution between $x = 0$ and $x = 4$?

In order to use the parametric form we must transform the integral to one over the variable t. So we note that

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = (4 - 2t)\,\mathrm{d}t = 2(2 - t)\,\mathrm{d}t$$

And the limits in x must be changed to limits in t. Using $x = 4t - t^2$ it is obvious that, for the lower limit:

$$x = 0 \quad \Rightarrow t = 0$$

Substituting the upper limit $x = 4$ in $x = 4t - t^2$ gives, for the upper limit:

$$t^2 - 4t + 4 = 0 \Rightarrow (t - 2)^2 = 0 \quad \Rightarrow t = 2$$

So the integral becomes:

$$V = \int_{t=0}^{t=2} \pi \left(\frac{1}{2}t^2\right)^2 2(2 - t)\,\mathrm{d}t = \frac{\pi}{2}\int_0^2 \left(2t^4 - t^5\right)\,\mathrm{d}t$$

Integrating, and applying limits gives:

$$V = \frac{\pi}{2}\left[\frac{2t^5}{5} - \frac{t^6}{6}\right]_0^2 = \frac{\pi}{2}(64)\left[\frac{1}{5} - \frac{1}{6}\right] = 32\pi\left[\frac{6 - 5}{30}\right] = \frac{16\pi}{15}$$

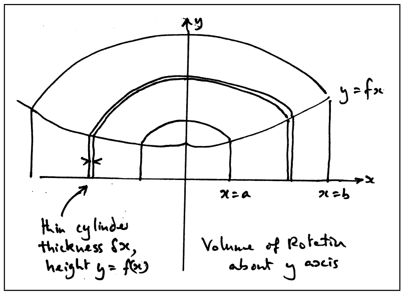


Figure: volume of revolution

6.2.4 Volume of Rotation – About y axis

Using the curve $y = f(x)$ another kind of volume can be defined. In this case the curve is rotated about the y axis, and then one can ask about the volume swept out by the area under the curve between $x = a$ and $x = b$ ($a < b$).

The volume has a lower surface defined by the plane $y = 0$, and upper surface defined by the rotation of the curve $y = f(x)$ about the y axis. Furthermore the volume has the form of a hollow cylinder aligned with the y axis, with $a^2 \leq x^2 + z^2 \leq b^2$ (where z is the third dimension).

The volume can be approximated by a set of thin cylindrical shell, radius x, height y, and with thickness δx. The volume δV of such an elemental cylindrical shell is the circumference ($2\pi x$) multiplied by its height and thickness. Summing all such elemental volumes:

$$V \approx \sum_{x=a}^{x=b} (2\pi x)(y)\delta x$$

As before, in the limit of many such elemental volumes the sum reduces to the integral:

$$V = \int_a^b 2\pi x y\,\mathrm{d}x$$

Example

Find volume of rotation about the y axis for the curve $y = 2x^2 + 3$ for $0 < x < 1$. Inserting directly into the above formula gives:

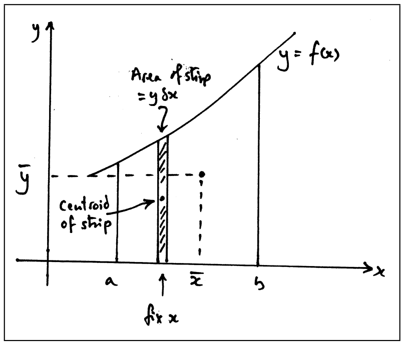
$$V = \int_0^1 2\pi x(2x^2 + 3)\,\mathrm{d}x$$

Integrating and applying limits gives:

$$V = 2\pi\left[\frac{2x^4}{4}+\frac{3x^2}{2}\right]_0^1 = 2\pi\left[\frac{1}{2}+\frac{3}{2}\right] = 4\pi$$

6.2.5 Centroid of a Plane Figure

[Like centre of a mass of a plane figure]



Consider the region in the x-y plane (i.e., plane figure) bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$ ($a < b$). The position of the centroid C $(\bar{x}, \bar{y})$ is defined in the following way. For the x position, consider the area as a set of strips parallel to the y axis. Each strip has area $y\delta x$. Then the product of the total area A and the x position of the centroid is (approximately) equal to the sum of the similar products for all the strips:

$$A\bar{x} \approx \sum_{x=a}^{x=b} x(y\delta x)$$

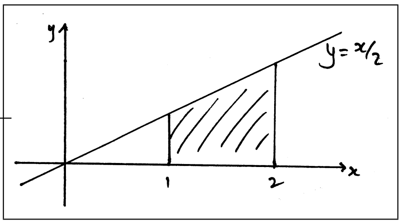
Note, that the rhs is the same as taking the total moment (about the y axis) of the area, assuming constant density. As the width of each strips decreases $\delta x \to 0$, and the sum approaches the integral:

$$A\bar{x} = \int_a^b xy \, \mathrm{d}x$$

And, of course $A = \int_a^b y \, \mathrm{d}x$.

The y position of the centroid, $\bar{y}$, can be found in a similar way. But now, one effectively takes the total moment about the x axis, and one uses the fact that for a rectangular strip of length y the moment of inertia about one end is $y/2$.

$$A\bar{y} = \int_a^b \frac{1}{2}y^2 \, \mathrm{d}x$$



Consider the position of the centroid of the plane figure formed by the region bounded by the lines $y = x/2$, $y = 0$, $x = 1$ and $x = 2$.

First, the total area of the firgure needs to be calculated:

$$A = \int_1^2 y \, \mathrm{d}x = \int_1^2 \frac{x}{2} \, \mathrm{d}x = \left[\frac{x^2}{4}\right]_1^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

Then, applying the above formulae:

$$A\bar{x} = \int_1^2 x\left(\frac{x}{2}\right) \, \mathrm{d}x = \frac{1}{2}\left[\frac{x^3}{3}\right]_1^2 = \frac{1}{6}\left[8-1\right]$$

Therefore,

$$\bar{x} = \frac{4}{3}\cdot\frac{7}{6} = \frac{14}{9} = 1\frac{5}{9}$$

Applying the similar process for the y position of the centroid:

$$A\bar{y} = \int_1^2 \frac{1}{2}y^2 \, \mathrm{d}x = \frac{1}{2}\int_1^2 \frac{x^2}{4} \, \mathrm{d}x = \frac{1}{8}\left[\frac{x^3}{3}\right]_1^2 = \frac{7}{24}$$

Therefore

$$\bar{y} = \frac{4}{3}\cdot\frac{7}{24} = \frac{7}{18}$$

6.2.6 Centre of Mass of Solid of Revolution

One can apply the same reasoning to find the position of the centroid of a solid of revolution formed by rotating the curve $y = f(x)$ about the x axis. The volume (again, for $a < x < b$) is approximated by a set of disks of thickness δx, radius y, and at a distance x from the y axis. Taking the moment about the y axis (assuming constant mass density):

$$V\bar{x} = \int_a^b x(\pi y^2)\,\mathrm{d}x$$

Where V is the total volume considered: $V = \pi \int_a^b y^2\,\mathrm{d}x$. The factors of π actually cancel out, so we can also write

$$\bar{x} = \frac{\int_a^b xy^2\,\mathrm{d}x}{\int_a^b y^2\,\mathrm{d}x}$$

Since the volume of revolution is symmetric about the plane $y = 0$, we have

$$\bar{y} = 0$$

Example

Find the x position of the centre of mass of a uniform solid hemisphere of radius a.

A solid hemisphere is formed from rotating the curve for a quarter of a circle of radius a: $x^2 + y^2 = a^2$ for $x > 0$ and $y > 0$. It follows that $y^2 = a^2 - x^2$. Applying the formula:

$$V\bar{x} = \int_0^a \pi x(a^2 - x^2)\,\mathrm{d}x = \pi\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{\pi a^4}{4}$$

The total volume is just half of that of a sphere, radius a:

$$V = \frac{1}{2}\cdot\frac{4}{3}\pi a^3$$

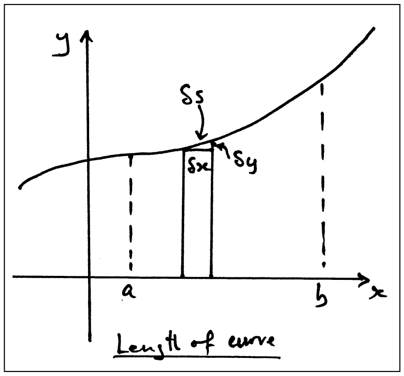
So,

$$\bar{x} = \left(\frac{2}{3}\pi a^3\right)^{-1}\frac{\pi a^4}{4} = \frac{3a}{8}$$

6.2.7 Length of Curve

Integration can be used to find the length of a curve. If a short line element of the curve is written as $\mathrm{d}s$, then, formally we can write that the total length is the integral over this line element:

$$S = \int \mathrm{d}s$$



Length of curve

But, since we are interested in the case where the curve is given by $y = f(x)$, we have to change this to an integral over x. In order to do so, we require $\frac{\mathrm{d}s}{\mathrm{d}x}$. This can be found by considering a small change in x of δx, which produces a corresponding change of δy in y. Now, the length of the line element as these changes happens is δs, and from Pythagoras we have

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

dividing by $(\delta x)^2$, and then letting $\delta x \to 0$ gives

$$\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 = 1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

And so,

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

Thus the total length of the curve between $x = a$ and $x = b$ is given by

$$S = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Example

Consider the curve $y = \cosh x$. Find the length along this curve from $x = 0$ to $x = 1$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh x \quad \Rightarrow 1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \sinh^2 x = \cosh^2 x$$

Therefore, applying the formula:

$$S = \int_0^1 \sqrt{\cosh^2 x}\,\mathrm{d}x = \int_0^1 \cosh x\,\mathrm{d}x = [\sinh x]_0^1 = \sinh 1 = \frac{1}{2}\left(e - e^{-1}\right)$$

Parametric Form

When a curve is given in parametric form, a similar formula for the length of the curve can be found. If $x = x(t)$ and $y = y(t)$, then we have, as before,

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

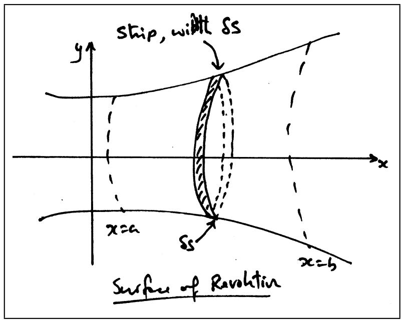
Dividing by $(\delta t)^2$ (the change in t corresponding to the change δx), and letting $\delta t \to 0$ gives

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}$$

So, we can transform the integral over the length of the curve to one over the parameter t:

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

6.2.8 Surface of Revolution



Now that we have an expression for a line element *along* a curve, we can find the area of the surface formed by rotating a curve $y = f(x)$ around the x axis, between $x = a$ and $x = b$. The continuous surface can be approximated by a set of circular strips of width δs, and radius y (at position x). Each strip has an area $\delta A = 2\pi y \delta s$. Dividing each side of this equation by δx and letting $\delta x \to 0$ we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2\pi y \frac{\mathrm{d}s}{\mathrm{d}x} = 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

where we have used the result from the last section. Integrating both sides with respect to x, we find:

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Example

Find the area generated by rotating, about the x axis, the parabola $y^2 = 8x$, between $x = 0$ and $x = 2$.

We have:

$$y = \sqrt{8x} = 2\sqrt{2x} \quad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{2} \cdot \frac{1}{2} x^{-1/2} \quad \Rightarrow \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{2}{x}$$

Applying the formula

$$A = \int_0^2 2\pi 2\sqrt{2}\sqrt{x}\sqrt{1 + \frac{2}{x}}\ \mathrm{d}x = 4\sqrt{2}\pi \int_0^2 \sqrt{x}\frac{(x+2)^{1/2}}{\sqrt{x}}\ \mathrm{d}x$$

Integrating, and applying limits:

$$A = 4\sqrt{2}\pi \left[\frac{(x+2)^{3/2}}{(3/2)}\right]_0^2$$

So,

$$A = \frac{8\sqrt{2}}{3}\pi(8 - 2\sqrt{2}) = \frac{8\pi}{3}(8\sqrt{2} - 4)$$

6.3 Numerical Approximate Integration

The fact that there is no single method to find the integral of an arbitrary function, means that there are many functions which it is difficult or impossible to find a functional form of their integral. But these functions might be well-behaved and integrable, ie their integral exists. In such cases it is possible to find an approximate value for the definite integral. This technique is extremely important in computational work.

A method which is rarely used is to expand the integrand in terms of a Taylor's series, and then integrate term by term. The more direct methods are essentially based on the approximation of the area under a curve as a sum of strips. All these methods have a precision which increases as the number of strips increases. The strips could be have different widths, but here we only consider constant width.

There are many different methods of numerical integration. They mainly differ in giving different precision (i.e., accuracy) for the same number of strips (or subintervals). For example, Simpson's Rule is more accurate than the Trapezoidal Rule, but it is slightly more complicated to calculate. Neither of the methods below are in widespread use in "real world" applications, but they are useful for understanding the methods used, for example, in computational fluid dynamics or computational electromagnetics, etc.

6.3.1 Trapezium Rule

In this method the idea is that each strip, which has an upper bound defined by the function, is replaced by a trapezoid. The area of each trapezoid is easy to calculated since it depends on the width of the strip and the value of the function at the edges of the trapezoid.

Consider the interval (a, b) divided uniformly, so that the x positions of the edges of the strips are $x_1, x_2, x_3, \ldots x_N$, where the spacing is $s = x_{k+1} - x_k$. The values of the function at these positions are $y_1, y_2, y_3, \ldots, y_N$, where $y_k = f(x_k)$. The area of the kth trapezoid is

$$A_k = \frac{s}{2}(y_k + y_{k+1})$$

So, the total area is the sum of all the trapezoids:

$$\begin{aligned} A &\approx \sum_k A_k \\ &\approx \frac{s}{2}((y_1+y_2)+(y_2+y_3)+(y_3+y_4)+\ldots+(y_{N-2}+y_{N-1})+(y_{N-1}+y_N)) \\ &\approx \frac{s}{2}(y_1+2y_2+2y_3+\ldots+2y_{N-1}+y_N) \end{aligned}$$

The pattern apparent in the last equation is summarized by what is known as *The Trapezium Rule:*

$$A \approx \frac{s}{2}(F+L+2R)$$

Where the terms are: F for first, L for last, and R for "the rest."

Example

Using the trapezium rule, integrate $\ln(1+x)$ between 0 and 0.75, using 5 equal intervals.

The spacing is $s = 0.75/5 = 0.15$, and the first x position is at zero. The following table can be constructed:

x_k	x value	$y_k = \ln(1+x_k)$	F	L	R
x_1	0.00	0.0000	0.0000		
x_2	0.15	0.13976			0.13976
x_3	0.30	0.26236			0.26236
x_4	0.45	0.37156			0.37156
x_5	0.60	0.47000			0.47000
x_6	0.75	0.55962		0.55962	

Notice that there are 6 values of x_k, but 5 subintervals (strips).

Using the Trapezium rule we have

$$A \approx \frac{0.15}{2}(0.0 + 2(1.2436) + 0.55962) = 0.22853$$

This example can also be done analytically, and gives an *exact* answer of 0.229328.

6.3.2 Simpson's Rule

Simpson's rule is a more accurate integration rule, given the same number of intervals. The number of strips has to be even (and the number of x positions odd).

$$A \approx \frac{s}{3}\left[(F+L)+4E+2R\right]$$

Here, the terms are: F: first, L: last, E: even numbered terms, R: the rest.

The idea behind Simpson's Rule is that pairs of strips define three points, and a quadratic can be found which passes through these three points. Then the approximate area of the strip is the area under this quadratic over the pair of strips. (This can be recognized from Taylor series, where a quadratic function can be found which is a better approximation than a linear function.)

Consider the points $x_1 = 0$, $x_2 = s$, $x_3 = 2s$, with corresponding values of the function: y_1, y_2, y_3. We have to find the quadratic with form $y = a + bx + cx^2$ that passes through these points, i.e., to find values for the constants a, b and c. Substituting $x_1 = 0$ we find $a = y_1$. The other two points give:

$$\begin{aligned} y_2 &= a + bs + cs^2 \\ y_3 &= a + 2bs + 4cs^2 \end{aligned}$$

Solving simultaneously:

$$y_3 - 4y_2 = -3a - 2bs \quad \Rightarrow b = \frac{1}{2s}(-3y_1 + 4y_2 - y_3)$$

And

$$c = \frac{1}{s^2}\left(y_2 - y_1 - \frac{1}{2}(-3y_1 + 4y_2 - y_3)\right) = \frac{1}{2s^2}(y_1 - 2y_2 + y_3)$$

The area under the quadratic can now be found:

$$\begin{aligned} A_{1,2} &= \int_0^{2s} (a + bx + cx^2)\, \mathrm{d}x = \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]_0^{2s} \\ &= 2as + 2bs^2 + \frac{8xs^3}{3} \\ &= s\left(2y_1 + (-3y_1 + 4y_2 - y_3) + \frac{4}{3}(y_1 - 2y_2 + y_3)\right) \\ &= \frac{s}{3}\,(y_1 + 4y_2 + y_3) \end{aligned}$$

Similarly the next pair of strips will give an area:

$$A_{3,4} = \frac{s}{3}\,(y_3 + 4y_4 + y_5)$$

Adding all such pairs of strips results in Simpson's Rule.

Example

Calculate the approximate value of the integral

$$\int_0^{\pi/3} \sqrt{\sin x}\, \mathrm{d}x$$

using Simpson's Rule with four equal intervals.

The spacing is $s = (1/4)(\pi/3) = \pi/12$.

k	x	$\sqrt{\sin x}$	$F + L$	E	R
1	0	0	0		
2	$\pi/12$	0.50874		0.50874	
3	$\pi/6$	0.70711			0.70711
4	$\pi/4$	0.84090		0.84090	
5	$\pi/3$	0.93060	0.93060		

So,

$$A = \frac{\pi}{9}\,((0 + 0.93060) + 4(0.50874 + 0.84090) + 2(0.70711)) = 0.6754$$

With 6 strips, Stroud finds 0.681, showing that one gets more significant digits as number of strips increases.