SUMMARY SHEET: DIFFERENTIATION
Engineering Maths II (MAE111), 2011

The fundamental definition: The **derivative** with respect to x of a function $y = f(x)$ is defined as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x}\right).$$

Notation: $\frac{\mathrm{d}y}{\mathrm{d}x}$, y', Dy, $\dot{y}$.

Function of a Function: For $y = f(u)$, with $u = g(x)$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Product Rule: For $u(x)$ and $v(x)$, the derivative of the **product** uv is:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Quotient Rule: Derivative of quotient (u/v):

$$D\left(\frac{u}{v}\right) = \frac{vDu - uDv}{v^2}$$

Implicit Differentiation: If $f(x, y) = 0$, then differentiate wrt to x, use fn. of fn. for any functions of y, and then rearrange for $\frac{\mathrm{d}y}{\mathrm{d}x}$. Example:

$$x^2 + y^2 = 1 \quad \Rightarrow \quad 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y}$$

(And substitute at end for y in terms of x, if possible.)

Parametric Differentiation: For $y = g(t)$, and $x = f(t)$, where t is the "parameter:"

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1}$$

Inverse Functions: If $x = f(y)$, then can write $y = f^{-1}(x)$ where f^{-1} is the inverse function of f. Example: If $x = \sin y$, then $y = \sin^{-1} x$, ie the inverse sine of x. *This is NOT the same as* $(\sin x)^{-1} = 1/(\sin x)$.

Differentiating Inverse Functions: If $y = f^{-1}(x)$ so $x = f(y)$, then differentiate wrt y, and ...

$$\frac{\mathrm{d}x}{\mathrm{d}y} = f'(y) \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$$

Logarithmic differentiation: If $y = f(x)g(x)h(x)$ (or any multiple product), take logs and differentiate implicitly to find $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\ln y = \ln f + \ln g + \ln h \quad \Rightarrow \quad \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{f}f' + \frac{1}{g}g' + \frac{1}{h}h'.$$

Derivatives to Remember!

$f(x)$	$D\,f(x)$
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$

$f(x)$	$D\,f(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

Rules of Logarithms:

$$\log A + \log B = \log AB \quad \text{and} \quad n \log A = \log A^n$$

Formula for change of base:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

The number e ...

$$e = 2.718281828\ldots$$

Important property: its slope at any point is equal to its value.

$$\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$$

Definition of Hyperbolic Functions

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

And also ...

$$\mathrm{cosech} x = \frac{1}{\sinh x}, \quad \mathrm{sech} x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}$$

Derivatives of Hyperbolic functions:

$$D(\sinh x) = \cosh x$$
$$D(\cosh x) = \sinh x$$
$$D(\tanh x) = \mathrm{sech}^2 x$$

Identities

$$\cosh^2 x - \sinh^2 x = 1$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

And many others. Similar to trigonometric identities with occasional change of sign. Prove by putting in terms of e^x and showing that RHS equal to LHS.

Solving equations For example,

$$3\cosh x - \sinh x = 3.$$

Put in terms of e^x, rearrange to solve for e^x, and then take logs.

Inverse Hyperbolic Functions For example, $\sinh^{-1} x$, and so on. Can be expressed in terms of logarithms, eg

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}).$$

To differentiate use same technique as given before for inverse functions.

Text Book References

	Mathematics for Engineers Croft & Davison	Engineering Mathematics Stroud (4th Edition)	Engineering Mathematics Stroud (5th Edition)
Differentiation	Ch 11, 12	Programmes 7.1 – 7.36, 9.1 – 9.13	Programme F10 Programme 7
Hyperbolic Fns	pp80-84	Programmes 3.1-3.30, 9.14-9.23	Programme 3

Table of Standard Integrals and Derivatives

(As given for Exams and Tests)

$f(x)$	$\int f(x)dx$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}, x > 0$	$\ln\|x\|$
e^{ax}	$\frac{1}{a}e^{ax}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln\|\cos x\| = \ln\|\sec x\|$
$\cot x$	$\ln\|\sin x\|$
$\sec x$	$\ln\|\sec x + \tan x\| = \ln\tan\|(\frac{\pi}{4} + \frac{1}{2}x)\|$
$\mathrm{cosec}\, x$	$-\ln\|\mathrm{cosec}\, x + \cot x\| = \ln\|\tan(\frac{1}{2}x)\|$
$\sec^2 x$	$\tan x$
$\mathrm{cosec}^2 x$	$-\cot x$
$\sec x \tan x$	$\sec x$
$\mathrm{cosec}\, x \cot x$	$-\mathrm{cosec}\, x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\mathrm{sech}^2 x$	$\tanh x$
$\mathrm{cosech}^2 x$	$-\coth x$
$\tanh x$	$\ln\cosh x$
$\coth x$	$\ln\|\sinh x\|$
$\mathrm{sech} x \tanh x$	$-\mathrm{sech} x$
$\mathrm{cosech} x \coth x$	$-\mathrm{cosech} x$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\frac{a+x}{a-x}, (x < a)$
$\frac{1}{\sqrt{(a^2-x^2)}}$	$\sin^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{(x^2+a^2)}}$	$\sinh^{-1}\left(\frac{x}{a}\right) = \ln\left[\frac{x+\sqrt{(x^2+a^2)}}{a}\right]$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right) = \ln\left[\frac{x+\sqrt{(x^2-a^2)}}{a}\right]$
$\frac{1}{x\sqrt{(x^2-a^2)}}$	$\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$