MAE111 Engineering Mathematics II (2003/2004 Sem. 1)

SECTION 2: DIFFERENTIATION

SECTION 3: HYPERBOLIC FUNCTIONS

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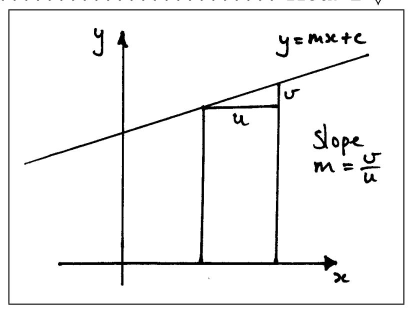
Section 2

Differentiation

⇓ .... *Hour 1* ⇓

2.1 Basic Definitions



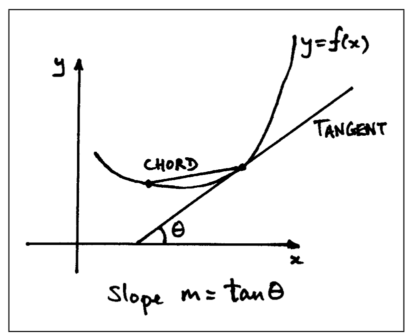
2.1.1 Slope of a Linear Function

The slope of a straight line is simply related to the coefficient of x in the linear function

$$y = f(x) = mx + c,$$

where m and c are constants. For two points on the line, distance u apart in x, with a separation of v in the y direction, the **slope** of the line is

$$m = \frac{v}{u}.$$

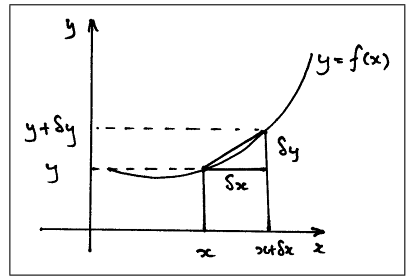


2.1.2 Slope of a function: The Tangent

We now want to extend our idea of a slope, so that we can use it to talk about the slope of any function, not just a straight line.

Consider a general function $y = f(x)$. One can draw a straight line through two points – this is a **chord**. As the two points are brought closer together, the change in the slope of the chord gets smaller and smaller. Eventually, when the two points are on top of each other, the line becomes the **tangent line**. This is the straight line passing through a point on the curve, with a slope equal to the slope of the function at that point. The slope is also equal to $\tan\theta$, where θ is the angle between the tangent and the x axis.

2.1.3 Derivative as slope of tangent line



For two points x and $x + \delta x$ we can calculate the slope of the chord for the function $y = f(x)$.

$$\frac{(y + \delta y) - y}{(x + \delta x) - x} = \frac{\delta y}{\delta x} \qquad \text{where } \delta y = f(x + \delta x) - f(x)$$

In the limit that $\delta x \to 0$ (i.e., δx "tends" to zero) this slope becomes equal to the slope of the tangent, ie the slope of the function at the point (x, y).

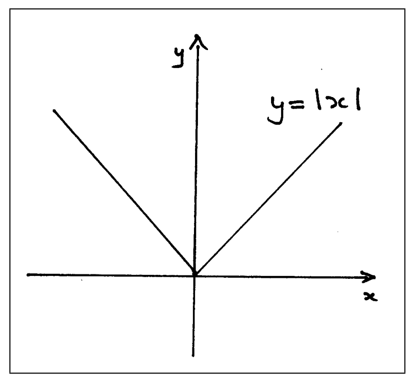
2.1.4 Definition

The **derivative** with respect to x of a function $y = f(x)$ is defined as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \to 0} \left(\frac{f(x + \delta x) - f(x)}{\delta x} \right).$$

We will use "wrt" for "with respect to" to indicate which variable is being used for the limit.

This definition makes it clear that the derivative of a function gives the slope of the tangent at a certain poin on the curve of that function. An alternative view is that the derivative gives the rate of change of the function.



2.1.5 Non-differentiable functions

Not all functions have a derivative at all points. Consider the function which is the modulus of x:

$$f(x) = |x|.$$

This has as derivative everywhere except at $x = 0$, where no sensible tangent can be defined.

2.1.6 Alternative notations

$$\frac{\mathrm{d}y}{\mathrm{d}x} \text{ or } \frac{\mathrm{d}}{\mathrm{d}x}(y) \text{ or } Dy \text{ or } f'(x) \text{ or } y'$$

2.1.7 Rate of change

If the independent variable is time t, instead of x, then the derivative gives the rate of change of a quantity, and sometimes the following notation is used:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y}$$

Examples: velocity is rate of change of position; acceleration is rate of change of velocity.

2.1.8 Higher order derivatives

Taking the derivative produces another function. So one can repeat the process:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f''(x) = D^2y = y''$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = f'''(x) = D^3y = y'''$$

Example

$$y = x^2 \qquad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x \qquad \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}(2x) = 2$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{\mathrm{d}}{\mathrm{d}x}(2) = 0$$

Example

$$y = \sin x \qquad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \cos x \qquad \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

2.2 Rules of Differentiation

The question we want to answer is: How do we find $\frac{\mathrm{d}y}{\mathrm{d}x}$ given $y = f(x)$ (or given an implicit function $F(x, y) = 0$)?

2.2.1 Standard results

The easiest way is to use your memory or a table of standard results!

Some derivatives to remember ...

$$\begin{aligned}
Dx^n &= nx^{n-1} \qquad \text{for some constant } n \\
De^x &= e^x \\
D\ln x &= \frac{1}{x} \\
D\sin x &= \cos x \\
D\cos x &= -\sin x \\
D\tan x &= \sec^2 x
\end{aligned}$$

2.2.2 Fundamental definition

Find $y'(x)$ for $y = x^3$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{(x+\delta x)^3 - x^3}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{1}{\delta x}(x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - x^3) \\
&= \lim_{\delta x \to 0}(3x^2 + 3x\delta x + (\delta x)^2)
\end{aligned}$$

$$= \quad 3x^2$$

2.2.3 Differentiation is a linear operation

If $y = f(x) + g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) + g'(x)$$

so it is easy to differentiate a polynomial function of x.

If $y = f(x)$ and k is a constant, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(ky) = \frac{\mathrm{d}}{\mathrm{d}x}[kf(x)] = k\frac{\mathrm{d}y}{\mathrm{d}x} = kf'(x).$$

$\Downarrow$ .... *Hour 2* $\Downarrow$

2.2.4 Chain rule (Function of a function)

Suppose that y is function of a variable u: $y = f(u)$. Also, u is itself a (different) function of x: $u = g(x)$. Then the derivative of y wrt to x is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Key Point

Use the chain rule for a function of a function.

"Differentiate y wrt u, and multiply by u differentiated wrt x."

Chain rule follows from fundamental definition, since

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta u}\frac{\delta u}{\delta x}\right)\\ &= \left(\lim_{\delta u\to 0}\frac{\delta y}{\delta u}\right)\left(\lim_{\delta x\to 0}\frac{\delta u}{\delta x}\right)\\ &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\end{aligned}$$

This follows since $\delta u = g(x+\delta x) - g(x)$ tends to zero as $\delta x \to 0$.

The chain rule can be applied even if the problem is not explicitly given in terms of two functions – just split it into two functions.

Example

$$y = \sin nx$$

Split as:

$$u = nx; \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = n; \qquad y = \sin u$$

Apply chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = n\cos u = n\cos nx$$

Example

$$y = (4 - x^2)^{1/3}$$

Example

$$y = \sin(\ln x)$$

Example

$$y = \ln(\sin x)$$

Example

$$y = \exp(\sin x)$$

Example

2.2.5 Product Rule and Quotient Rule

For two functions of x, $u(x)$ and $v(x)$, the derivative of the **product** uv is:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Key Point

Product Rule:
$D(uv) = uDv + vDu.$

The derivative of the **quotient** u/v is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}\right)$$

The quotient rule might be easier to remember using D notation:

$$D\left(\frac{u}{v}\right) = \frac{vDu - uDv}{v^2}$$

The quotient rule can be found from the product rule, using the chain rule:

$$D\left(\frac{u}{v}\right) = uD\left(\frac{1}{v}\right) + \frac{1}{v}Du = -\frac{u}{v^2}Dv + \frac{1}{v}Du$$

Example

$$y = x\ sinx$$

$$u = x; \qquad v = \sin x; \qquad Du = 1; \qquad Dv = \cos x$$

$$Dy = D(uv) = x\cos x + \sin x$$

Example

Find the derivate of

$$y = \frac{e^x}{x}$$

Treat this as product so that $u = e^x$ and $v = x^{-1}$

$$\begin{aligned} Dy &= uDv + vDu = e^x\left(\frac{-1}{x^2}\right) + \left(\frac{1}{x}\right)e^x \\ &= \frac{e^x}{x^2}(x-1) \end{aligned}$$

Example

$$y = \tan x = \frac{\sin x}{\cos x}$$

$$\begin{aligned} Dy &= D\left(\frac{u}{v}\right) = \frac{vDu - uDv}{v^2} \\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

Example

$$y = \sec x = \frac{1}{\cos x}$$

Example

$$y = e^x \cos(e^x)$$

Example

$$y = \ln[(x+1)(x+2)(x+3)]$$

Using rules of logarithms:

$$\begin{aligned} y &= \ln(x+1) + \ln(x+2) + \ln(x+3) \\ Dy &= \frac{1}{x+1}D(x+1) + \frac{1}{x+2}D(x+2) + \frac{1}{x+3}D(x+3) \\ &= \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} \end{aligned}$$

2.2.6 Implicit Differentiation

A function is a relationship between variables. In the simple case of a function of one variable, there is an independent variable (usually written x), and the value of this variable determines the value of the dependent variable, ie the function value. If y is a function of x, then we write $y = f(x)$. If we write this function so that only terms containing x are on the left hand side, then we say that it is an **explicit** function of x – in other words y is given completely in terms of x. For example

$$y = \sqrt{1 - x^2}$$

is an explicit function of x. On the other hand we can square and rearrange to produce:

$$x^2 + y^2 = 1$$

This still defines a relationship between x and y, but we now say that y is an **implicit** function of x, since y is not given in terms of x alone. Instead there is an equality involving both x and y. We can differentiate this implicit function wrt x, by using the chain rule on any terms containing y.

Key Point

For implicit functions use the chain rule for terms containing y.

So, for example, differentiating wrt x both sides of the equation, and rearranging to solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$x^2 + y^2 = 1 \qquad \Rightarrow 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y}$$

We can replace y by its explicit form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -x(1-x^2)^{-1/2}$$

Note the same result can be obtained in this case by using the chain rule on the explicit form of the function:

$$y = (1-x^2)^{1/2} \qquad \Rightarrow y' = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = -x(1-x^2)^{-1/2}$$

But often one *cannot* find the explicit form of a function, so one has to differentiate implicitly.

Example

$$x^3 + y^3 = 3xy$$

(Cannot find y explicitly.) Note that rhs is a product.

$$\begin{aligned} 3x^2 + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 3y.(1) + 3x\frac{\mathrm{d}y}{\mathrm{d}x} \\ \frac{\mathrm{d}y}{\mathrm{d}x}\left(3y^2 - 3x\right) &= 3y - 3x^2 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y-x^2}{y^2-x} \end{aligned}$$

Note that cannot substitute for y on rhs.

Example

Calculate y' and y'' for the implicit function

$$x^2 + y^2 = 1$$

Example

If $xe^y = \cos y$, calculate y' when $x = 1$ and $y = 0$.

(NB: For this kind of problem the chosen values of x and y must satisfy the implicit function – and it is always a good idea to check if they do!)

$\Downarrow$ .... *Hour 3* $\Downarrow$

2.2.7 Logarithmic Differentiation

This techique is useful when a function is a product of many functions, or when a function of x appears in a power. Starting from $y = f(x)$, the idea is to take logs of both sides and differentiate implicitly.

Example

$$y = x^3 e^x \sin x$$

Take logs, and differentiate using chain rule as in implicit differentiation:

$$\begin{aligned} \ln y &= 3\ln x + x + \ln(\sin x) \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{x} + 1 + \frac{\cos x}{\sin x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \left(\frac{3}{x} + 1 + \frac{\cos x}{\sin x}\right) x^3 e^x \sin x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= e^x x^2\left((3+x)\sin x + x\cos x\right) \end{aligned}$$

Example

$$y = (x+1)(x+2)(x+3)(x+4)$$
$$\ln y = \ln(x+1) + \ln(x+2) + \ln(x+3) + \ln(x+4)$$

Example

For product of several functions of x:

$$y = u(x).v(x).w(x).z(x)$$

Example

$$y = a^x$$

where a is a constant.

$$\begin{aligned} \ln y &= x \ln a \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln a \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y \ln a = a^x \ln a \end{aligned}$$

Example

$$y = x^x$$

Take logs and use product rule on rhs:

$$\begin{aligned} \ln y &= x \ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= x.\left(\frac{1}{x}\right) + (1).\ln x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y(\ln x + 1) = x^x(\ln x + 1) \end{aligned}$$

Example

$$y = (\cos x)^{\sin x}$$

2.3 Parametric Equations

A function can be thought of as a relationship between two variables. So far we have considered that relationship as expressed by a single function. However, suppose that the two variables are functions of a third variable: consider $x = f(t)$ and $y = g(t)$, where x and y are functions of a **parameter** t. Each value of t gives a value for x and a value for y. Then x and y are related by the value of the parameter. For example:

$$x = \cos t, \qquad y = \sin t \qquad \text{or} \qquad x = t^2, \qquad y = 2t.$$

In these cases we say that x and y are given in **parametric form**.

A common example of parametric form is when the x and y coordinates of some object are given as functions of time t.

Sometimes, the two equations containing t can be combined to eliminate t, and then one finds a direct relationship between x and y (either explicit or implicit). For example,

$$x = \cos t, \qquad y = \sin t \qquad \Rightarrow x^2 + y^2 = 1$$

and

$$x = t^2, \qquad y = 2t \qquad \Rightarrow y^2 = 4x$$

We can use these forms to calculate $y'(x)$.

However, sometimes it is impossible to eliminate t. For example:

$$x = t - \sin t, \qquad y = 1 - \cos t \qquad \text{or} \qquad x = t + t^3, \qquad y = t - t^4.$$

We can differentiate in this case, by starting from $y = g(t)$, and by assuming that t is a function of x (in fact, $t = f^{-1}(x)$), and using the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} = \frac{\dot{y}}{\dot{x}}$$

Note we have used the dot notation for derivates wrt t.

Incidentally, when Newton invented his method of differentiation, he considered functions in parametric form, and introduced the dot notation.

Example

Consider

$$x = \cos t, \qquad y = \sin t$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{\cos t}{-\sin t} = -\cot t$$

To find higher derivatives takes some care:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1}$$

It might be convenient to use this directly, or, to substitute for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}}{\dot{x}}\right)(\dot{x})^{-1} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

where the quotient rule has been used.

Note that

$$y'' \neq \frac{\ddot{y}}{\ddot{x}}$$

Example

Calculate y'' for $x = t^2$, $y = 2t$.

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{2}{2t} = \frac{1}{t}$$

$$y'' = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\right)\frac{1}{\dot{x}} = \frac{-t^{-2}}{2t} = \frac{-1}{2t^3}$$

Example

Calculate y' and y'' for $x = \sin t$, $y = \cos 2t$.

2.4 Differentiating Inverse Functions

2.4.1 Inverse Functions

Suppose that x is given as a function f of y, i.e.,

$$x = f(y).$$

(This is the other way around from usual!) This means that given the function f tells what value of x corresponds to a given value of y. Now, the question is what function will tell us the value of y given a value of x?

So, we suppose that there is a function g which gives the value of y corresponding to a given value of x:

$$y = g(x)$$

So we can write:

$$y = g(\, f(y)\,)$$

In other words, the function g "undoes" the effect of the function f. The function that does this is called the **inverse function**, and is written

$$y = f^{-1}(x).$$

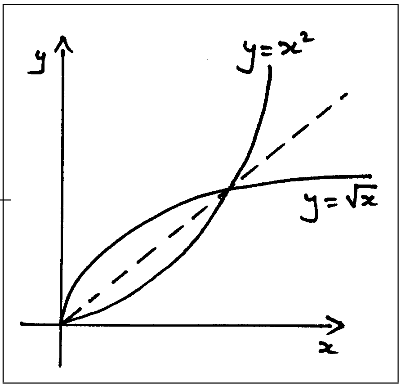
Key Point

The notation for the inverse function $f^{-1}(x)$ is NOT the same as "power to the minus one" or "one over f."

We say that f^{-1} is the inverse function of f. The superscript does *not* mean raise to the power -1. The same notation is used for the inverse trig functions such as $\sin^{-1}$.

Since the inverse function reverses the effect of a function, we can see that, then using the inverse function of a function results in no change, and vice versa.

$$y = f^{-1}(\, f(y)\,) = f(\, f^{-1}(y)\,)$$



If $x = f(y) = y^2$, then we can rewrite as $y = f^{-1}(x) = \sqrt{x}$.

Remember, that a function can operate using any variable, so we can write these two functions as functions of x.

$$\begin{aligned} y &= f(x) = x^2 \qquad (x \geq 0) \\ y &= f^{-1}(x) = \sqrt{x} \qquad (x \geq 0) \end{aligned}$$

Note, that the graphs of these two functions are mirror images of each other in the line $y = x$.

Note also, that in the above example we had to restrict the range of x so that the inverse function exists.

Example

If $x = e^y$, then $y = \ln x$, so the natural logarithm is the inverse function to the exponential function.

$$\exp^{-1}(x) = \ln x$$

Example

Find the inverse function of $y = f(x) = x^2 + 3$.

Rewriting:

$$x^2 = y - 3 \qquad \Rightarrow x = \sqrt{y-3} = g(y)$$

Changing the role of x and y around then gives

$$y = f^{-1}(x) = g(x) = \sqrt{x-3}$$

We can try this out: For $x = 2$, $y = f(2) = 4 + 3 = 7$. And then reverse the process: $x = 7$, $y = f^{-1}(7) = \sqrt{7-3} = 2$.

$\Downarrow$ . . . . *Hour 4* $\Downarrow$

2.4.2 Differentiating inverse functions

Start from $y = f^{-1}(x)$. Operating on both sides with f and differentiating with respect to y gives

$$x = f(y) \qquad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = f'(y)$$

We use the following important result:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

This is not not quite the same as manipulating fractions, but looks like it!

So to find the derivative of the inverse function:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{f'(y)}$$

This is in terms of y, and it can be put in terms of x using the inverse function itself:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{f'[\,f^{-1}(x)\,]}$$

Example

If $y = \ln x$, then $x = e^y$, ie by taking the inverse. Then differentiate wrt y.

$$\frac{\mathrm{d}x}{\mathrm{d}y} = e^y$$

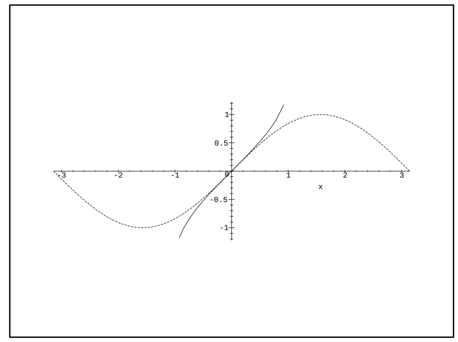
and finally switch over...

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^y} = \frac{1}{x}$$

2.4.3 Inverse Trigonometric Functions

If $x = \sin y$, then $y = \sin^{-1} x$. *(This is not the same as* $1/(\sin x)$.*)* But for any given value of x ($-1 \le x \le +1$) there are an infinite number of values of y. For example, if $x = 1/2$, then the possible values are

$$x = \frac{1}{2}, \qquad y = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, \ldots$$



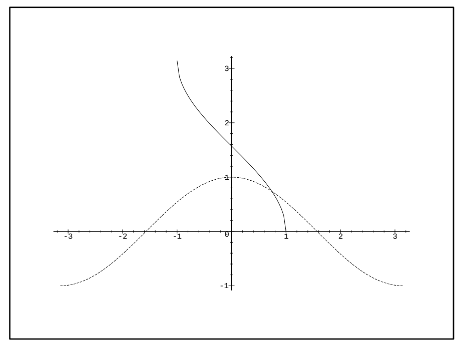
$\sin x$ and $\sin^{-1} x$

To restrict the inverse trigonometric to a single value, we define the **principal value**, as the value of the inverse function in a certain interval in y.

Function	Principal value interval
$\sin^{-1} x$	$-\pi/2 \le y \le \pi/2$
$\cos^{-1} x$	$0 \le y \le \pi$
$\tan^{-1} x$	$-\pi/2 \le y \le \pi/2$

For example, for the function $\sin^{-1} x$ we mean "the angle between $-\pi/2$ and $\pi/2$ whose sine is x."

2.4.4 Derivative of $\sin^{-1} x$



$\cos x$ and $\cos^{-1} x$

Let

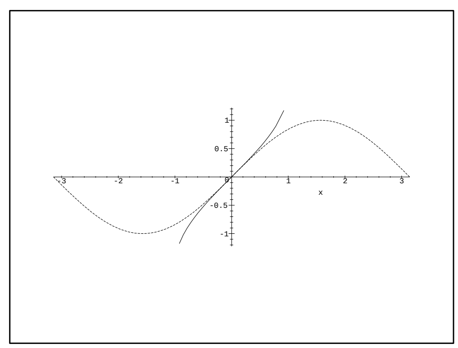
$$y = \sin^{-1} x; \qquad \Rightarrow x = \sin y \qquad \Rightarrow \frac{dx}{dy} = \cos y$$

Using results from earlier section:

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{\pm 1}{\sqrt{1 - \sin^2 y}} = \frac{\pm 1}{\sqrt{1 - x^2}}$$

(Remember: $\sin^2 y + \cos^2 y = 1$.) And for principal value we can show (eg from graph of principal value) that only the positive term is required:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$$



$\tan x$ and $\tan^{-1} x$

2.4.5 Derivative of $\cos^{-1} x$

Let

$$y = \cos^{-1} x; \qquad \Rightarrow x = \cos y \qquad \Rightarrow \frac{dx}{dy} = -\sin y$$

Therefore,

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{\mp 1}{\sqrt{1 - \cos^2 y}} = \frac{\mp 1}{\sqrt{1 - x^2}}$$

And for principal value we can show (eg from graph of principal value) that only the negative term is required:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}$$

2.4.6 Derivative of $\tan^{-1} x$

Let

$$y = \tan^{-1} x; \qquad \Rightarrow x = \tan y = \frac{\sin y}{\cos y} \qquad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = \sec^2 y$$

Therefore,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 x$$

Now, taking $\cos^2 y + \sin^2 y = 1$, and dividing by $\cos^2 y$:

$$1 + \tan^2 y = \frac{1}{cos^2 y} \qquad \Rightarrow \cos^2 y = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

Therefore,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 + x^2}$$

Section 3

Hyperbolic Functions

⇓ .... *Hour 5* ⇓

3.1 The Exponential function

There is this number called e ...

$$e = 2.718281828\ldots$$

The number e is as important as π, but not as well known. The origin of π lies in the properties of the circle (area πr^2, circumference $2\pi r$, etc.) The number e can be related to the properties of the hyperbola, but also from a unique property associated with differentiation.

Functions of the form

$$f(x) = b^x$$

where b is a constant, are known as exponential functions. For example, consider 2^x for integral x:

2^{-4}	2^{-3}	2^{-2}	2^{-1}	2^0	2^1	2^2	2^3
$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

One can see that x increases the functions becomes ever larger, and as x decreases the function becomes ever smaller. There are no maxima or minima.

The derivative of an exponential functions can be found using the fundamental definition of differentiation.

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{\delta x \to 0}\left(\frac{b^{x+\delta x} - b^x}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{b^x b^{\delta x} - b^x}{\delta x}\right) \\ &= b^x \lim_{\delta x \to 0}\left(\frac{b^{\delta x} - 1}{\delta x}\right) \end{aligned}$$

The ratio on the rhs is an **indeterminate** form, i.e., as $\delta x \to 0$, the ratio appears to equal $\frac{0}{0}$.

In such cases, as discussed later, it is not obvious that a limit even exists. However, it can be shown that in this case a limit does exist, and so we can assume that it is equal to some constant value, say k, i.e.,

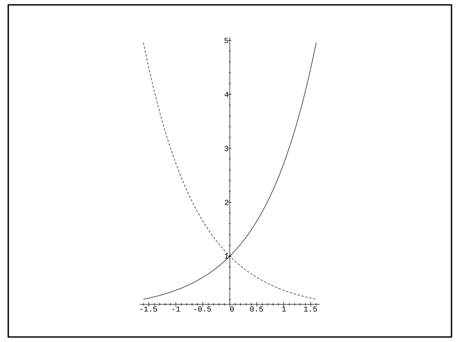
$$k = \lim_{\delta x \to 0}\left(\frac{b^{\delta x} - 1}{\delta x}\right)$$

It follows then that

$$f(x) = b^x \qquad \Rightarrow \frac{\mathrm{d}f}{\mathrm{d}x} = k\, b^x$$

So, the derivative of an exponential function is equal to the same function multiplied by some constant value. It is clear, that we can now choose the value of b to make the constant of proportionality equal to one, i.e., $k = 1$. This special, unique, value for b is called the number e, and it is the precise number such that the derivative of the function e^x is equal to itself.

$$f(x) = e^x \qquad \Rightarrow \frac{\mathrm{d}f}{\mathrm{d}x} = \; e^x$$



The fact that there is only one such number that has this property makes it very special, and a "natural" choice. So natural, that this is the reason why logarithm to base e, $\log_e$, is called the natural logarithm, $\ln$. The function e^x is called *the* exponential function, to distinguish it from all other exponential functions.

The exponential function is the solution to the following differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f \qquad \Rightarrow f = Ce^x$$

There are many situations in the real world where the rate of change of a quantity is equal to the quantity itself. Consider, the population of rabbits: Two rabbits make two baby rabbits (4 total). In turn all four produce 4 baby rabbits (8 total), and so on. So this rabbit population increases exponentially. (Many rabbits very quickly!) The decay of a radioactive material is proportional to the amount remaining, so that radioactivity decreases exponentially.

As another example: The rate of discharge of charge stored in a capacitor is proportional to the amount of charge remaining. So if the charge is $Q(t)$, and at $t = 0$ the charge is Q_0, then

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ \qquad \Rightarrow Q = Q_0 e^{-kt}$$

The value $\tau = 1/k$ is the time constant for the discharge.

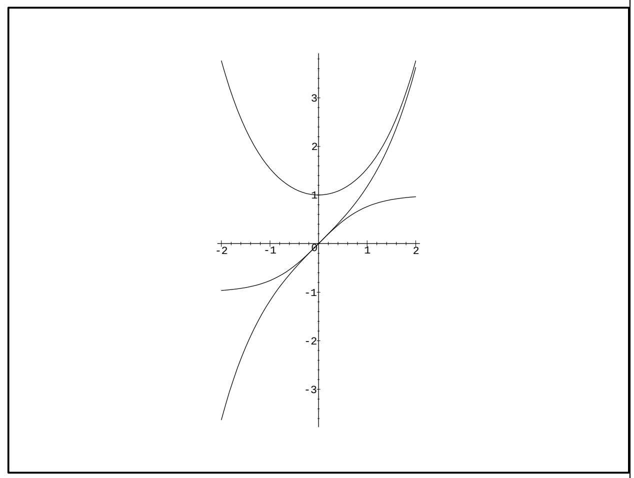
3.2 Hyperbolic Functions

3.2.1 Definitions

The hyperbolic functions are formed by combinations of exponentials

$$\begin{aligned}\sinh x &= \frac{1}{2}\left(e^x - e^{-x}\right)\\ \cosh x &= \frac{1}{2}\left(e^x + e^{-x}\right)\\ \tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}\end{aligned}$$

These functions are pronounced in different ways, eg the hyperbolic sine function $\sinh x$ is pronounced "shine" x, or "sinch" x.



Note from the graphs for these functions, that since $D \sinh x = \cosh x > 0$ the graph of $\sinh x$ always has a positive slope, so is continually increasing for $x > 0$. Similar arguments apply to $\cosh x$.

Other hyperbolic functions follow from similarity with the trigonometric functions:

$$\text{cosech}x = \frac{1}{\sinh x}, \qquad \text{sech}x = \frac{1}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

Note from the definitions:

$$\begin{aligned}\sinh(-x) &= \frac{1}{2}\left(e^{-x} - e^x\right) = -\sinh x\\ \cosh(-x) &= \frac{1}{2}\left(e^{-x} + e^x\right) = \cosh x\\ \tanh(-x) &= \frac{\sinh(-x)}{\cosh(-x)} = -\tanh x\end{aligned}$$

We can also use the definitions to solve for e^x and e^{-x}:

$$\begin{aligned}e^x &= \cosh x + \sinh x\\ e^{-x} &= \cosh x - \sinh x\end{aligned}$$

We can also notice that

$$e^x e^{-x} = (\cosh x + \sinh x)(\cosh x - \sinh x) = \cosh^2 x - \sinh^2 x = 1$$

Key Point

$\cosh$ *and* $\sinh$ *are related by*

$$\cosh^2 x - \sinh^2 x = 1$$

3.2.2 Derivatives of hyperbolic functions

Using the basic definitions it is easy to calculate the derivatives. Consider

$$y = \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \frac{1}{2}\left(e^x - (-1)e^{-x}\right) = \cosh x$$

Similarly for $y = \cosh x = \frac{1}{2}\left((e^x + e^{-x})\right)$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x$$

Note that there is no change of sign, as for the derivative of cosine.

The derivative of hyperbolic tangent can be found using the quotient rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sinh x}{\cosh x}\right) \\
&= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} \\
&= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} \\
&= \text{sech}^2 x
\end{aligned}$$

Note that there are similarities with the behaviour of the trigonometric functions, but sometimes with a change of sign.

3.2.3 Representation as series

Starting from the series for the exponential function:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

So, from the definition of $\sinh x$

$$\begin{aligned}
\sinh x &= \frac{1}{2}\left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)\right\} \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots
\end{aligned}$$

Similarly,

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

3.2.4 Identities

A number of identities can be proved involving the hyperbolic functions, and these are similar to trigonometric identities.

We have already seen that

$$\cosh^2 x - \sinh^2 x = 1$$

Similarly, dividing by $\cosh^2 x$,

$$\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = 1 - \tanh^2 x = \text{sech}^2$$

and also

$$\frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} = \coth^2 x - 1 = \text{cosech}^2$$

We can also prove the following equivalents to the trigonometric double angle identities.

$$\begin{aligned}
\cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \\
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\
\cosh 2x &= \cosh^2 x + \sinh^2 x \\
\sinh 2x &= 2 \sinh x \cosh x
\end{aligned}$$

As an example of proving such identities, consider

$$\begin{aligned}
\cosh(x+y) &= \frac{1}{2}(e^{x+y} + e^{-x-y}) = \frac{1}{2}(e^x e^y + e^{-x} e^{-y}) \\
&= \frac{1}{2}\left[(\cosh x + \sinh x)(\cosh y + \sinh y) + (\cosh x - \sinh x)(\cosh y - \sinh y)\right] \\
&= \frac{1}{2}\left[\cosh x \cosh y + \cosh x \sinh y + \sinh x \cosh y + \sinh x \sinh y \right. \\
&\quad \left. + \cosh x \cosh y - \cosh x \sinh y - \sinh x \cosh y + \sinh x \sinh y\right] \\
&= \cosh x \cosh y + \sinh x \sinh y
\end{aligned}$$

which completes the proof.

$\Downarrow$ . . . . *Hour 6* $\Downarrow$

3.2.5 Solving Equations

Consider an equation of the form

$$a \sinh x + b \cosh x = c$$

and suppose that we must solve for x, given values for the constants a, b, and c. Equations of this form can be expressed as a quadratic equation in e^x, and solved for e^x, and then taking logs to find x.

Example

$$3 \cosh x - \sinh x = 3$$

Using definitions:

$$\frac{3}{2}\left(e^x + e^{-x}\right) - \frac{1}{2}\left(e^x - e^{-x}\right) = 3$$

$$e^x + 2e^{-x} = 3$$

and multiplying by e^x gives

$$e^{2x} - 3e^x + 2 = 0$$

which can be written as

$$X^2 - 3X + 2 = 0$$

where $X = e^x$. The lhs factorises, to give $(X-1)(X-2) = 0$, so

$$e^x = 1 \qquad \text{or} \qquad e^x = 2$$

that is:

$$x = 0 \qquad \text{or} \qquad x = \ln 2$$

Note that for real x, $e^x > 0$, so if $X < 0$ then there can be no solution for x.

3.2.6 Inverse Hyperbolic functions

We know that the log function $\ln x$ is the inverse function to the exponential e^x. So, the question arises: can we can relate the inverse hyperbolic functions to logarithms

Suppose that $x = \sinh y$, then we can define the inverse hyperbolic sine function: $y = \sinh^{-1} x$.

The inverse hyperbolic functions can be expressed in terms of logs. Let us consider

$$y = \sinh^{-1} x$$

So:

$$x = \sinh y = \frac{1}{2}(e^y - e^{-y})$$

Multiply by $2e^y$ and rearrange:

$$e^{2y} - 2xe^y - 1 = 0$$

which is a quadratic in $Y = e^y$

$$Y^2 - 2xY - 1 = 0$$

and which can be solved for Y using the standard formula

$$Y = e^y = \frac{1}{2}\left[2x \pm \sqrt{4x^2 + 4}\right] = x \pm \sqrt{x^2 + 1}$$

Now, $\sqrt{x^2 + 1} > x$, so we drop the minus sign to keep $Y > 0$ (which is required so that y is real). And finally taking logs:

$$y = \sinh^{-1} x = \ln[x + \sqrt{x^2 + 1}]$$

A similar process can be carried out for

$$y = \cosh^{-1} x$$

So:

$$x = \cosh y = \frac{1}{2}(e^y + e^{-y})$$

Multiply by $2e^y$ and rearrange:

$$e^{2y} - 2xe^y + 1 = 0$$

which is a quadratic in $Y = e^y$

$$Y^2 - 2xY + 1 = 0$$

and which can be solved for Y using the standard formula

$$Y = \frac{1}{2}\left[2x \pm \sqrt{4x^2 - 4}\right]$$

This has two possible solutions:

$$x + \sqrt{x^2 - 1} \qquad \text{or} \qquad x - \sqrt{x^2 - 1}$$

Now it turns out that $(x - \sqrt{x^2 - 1})$ is equal to $(x + \sqrt{x^2 - 1})^{-1}$, for the following reason:

$$x - \sqrt{x^2 - 1} = x - \sqrt{x^2 - 1}\left(\frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}\right) = \frac{x^2 - (x^2 - 1)}{x + \sqrt{x^2 - 1}} = \frac{1}{x + \sqrt{x^2 - 1}}$$

So,

$$e^y = x + \sqrt{x^2 - 1} \qquad \text{or} \qquad e^y = \left(x + \sqrt{x^2 - 1}\right)^{-1}$$

Taking logs gives

$$y = \cosh^{-1} x = \pm \ln\left(x + \sqrt{x^2 - 1}\right)$$

Note that $\cosh^{-1} x$ only exists if $x \geq 1$ (see graph of $\cosh$).

Finally, we can express $y = \tanh^{-1} x$ in terms of logs.

$$x = \tanh y = \frac{(e^y - e^{-y})}{(e^y + e^{-y})}$$

$$xe^y + xe^{-y} = e^y - e^{-y}$$

$$(1 - x)e^y = (1 + x)e^{-y}$$

$$e^{2y} = \frac{(1 + x)}{(1 - x)}$$

Leading to:

$$y = \tanh^{-1} x = \frac{1}{2} \ln\left[\frac{(1 + x)}{(1 - x)}\right]$$

Note that $y = \tanh^{-1} x$ only exists if $-1 \leq x \leq +1$ which is required so that it is not the log of a negative quantity.

Summary: Inverse Hyperbolics as Log Functions

$\sinh^{-1} x$	$\ln[x + \sqrt{x^2 + 1}]$	
$\cosh^{-1} x$	$\pm \ln[x + \sqrt{x^2 - 1}]$	$(x \geq 1)$
$\tanh^{-1} x$	$\frac{1}{2} \ln\left[\frac{1+x}{1-x}\right]$	$(-1 \leq x \leq +1)$

3.3 Differentiation of Inverse Hyperbolic Functions

Differentiation of the inverse functions is similar to other such functions, ie invert, differentiate and use $\frac{dy}{dx} = 1/(\frac{dx}{dy})$.

Consider $y = \sinh^{-1} x$, so

$$x = \sinh y \qquad \Rightarrow \frac{dx}{dy} = \cosh y$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{x^2 + 1}}$$

Consider $y = \cosh^{-1} x$, so

$$x = \cosh y \qquad \Rightarrow \frac{dx}{dy} = \sinh y$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

Consider $y = \tanh^{-1} x$, so

$$x = \tanh y \qquad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = \text{sech}^2 y$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\text{sech}^2 y} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2}$$

These results can also be found from the log form for the inverse functions. For example:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \sinh^{-1} x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x + \sqrt{x^2+1})\right] \\
&= \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right) \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

Summary: Differentiation of Inverse Hyperbolic Functions

y	$\frac{\mathrm{d}y}{\mathrm{d}x}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

3.4 Examples of Differentiating Hyperbolic functions

Example

Function of a function (ie chain rule):

$$D \sinh(x^2 + x + 1) = (2x + 1)\cosh(x^2 + x + 1)$$

Example

Product rule (and chain rule):

$$D(e^x \cosh 3x) = e^x 3 \sinh 3x + e^x \cosh 3x = e^x(3 \sinh 3x + \cosh 3x)$$

Example

Quotient rule:

$$\begin{aligned}
D\text{sech}x &= D\left(\frac{1}{\cosh x}\right) \\
&= \frac{\cosh x.(0) - (1).\sinh x}{\cosh^2 x} = \frac{-\sinh x}{\cosh^2 x} \\
&= -\text{sech}x \tanh x
\end{aligned}$$

Example

$$D(\ln x \tanh x) = \ln x \text{sech}^2 x + \frac{1}{x}\tanh x$$

Example

$$D\tanh^{-1}(\tan x) = \frac{1}{1-\tan^2 x}D(\tan x) = \frac{\sec^2 x}{1-\tan^2 x}$$

Example

$$D\cosh^{-1}(e^{2x}) = \frac{1}{\sqrt{e^{4x}-1}}2e^{2x}$$