MAE111 Engineering Mathematics II (2003/2004 Sem. 1)

SECTION 8: COMPLEX NUMBERS

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December 9, 2003

Section 8

Complex Numbers

8.1 Using our Imagination

Think of a number…

$$81$$

Now, what number multiplied by itself will make this number?

$$9 \times 9 = 81$$

We say that 9 is the square root of 81.

$$\sqrt{81} = 9$$

But it is clear that also

$$(-9) \times (-9) = 81$$

So, in fact there appear to be two square roots:

$$\sqrt{81} = \pm 9$$

We have stretched the idea of square roots to include negative square roots, but… *what about the square root of a negative number?*

$$\sqrt{-16} = ?$$

Well, we can rewrite this as

$$\sqrt{-16} = \sqrt{(-1)(16)} = \sqrt{16}\ \sqrt{-1} = 4\sqrt{-1}$$

So, the square root of *any* negative number can be written, apparently, in terms of the square root of (-1). But what is the square root of (-1)?

It would seem obvious that to get a square of size unity, we should start from unity itself, but $(+1)^2 = +1$ and $(-1)^2 = +1$, so it is (*apparently*) impossible to find a number which is the square root of (-1).

8.1.1 The Number j

But, let us suppose that there was such a number, but that it was completely different from all the other numbers which we usually deal with, such that we had to keep account of it separately. This is like a leap of imagination: we suppose there is such a number and then we see if can make a logical, mathematical, set of rules for this number.

$$j = \sqrt{-1}$$

(In maths and physics, sometimes i is used instead of j. Engineers use j to avoid confusion with i for current.)

Just from this definition we can find all the powers of j in terms of j and 1:

$$j^2 = -1, \quad j^3 = j^2 j = -j, \quad j^4 = (j^2)^2 = (-1)^2 = +1, \quad j^5 = j^4 j = j$$

$$j^{-1} = \frac{1}{j} = \frac{j}{j^2} = -j, \quad j^{-2} = \frac{1}{j^2} = -1, \quad j^{-3} = \frac{1}{j^2}\frac{1}{j} = j, \quad j^{-4} = j^{-2}j^{-2} = +1$$

. . . and so on.

8.1.2 Imaginary, Real, and Complex Numbers

We can now write

$$\sqrt{-16} = j4$$

A number which includes j is called an **imaginary** number. To make the distinction with the "ordinary" numbers these are called the **real** numbers. A real number multiplied by j produces an imaginary number.

The terms "real" and "imaginary" are basically useful names. The real numbers can be used to count "real" things, but the imaginary numbers are just as "real" to a mathematician. And we will see how even physical quantities can sometimes be represented using imaginary numbers.

We want to build up a set of rules which uses the number j. We already have multiplication of j by a real number. We can also try adding a real number and an imaginary number:

$$10 + j5$$

This is neither wholly real, or wholly imaginary. We call such numbers: **complex numbers**.

A complex number is written

$$z = a + jb$$

where a is the *real part* of the complex number, and b is the *imaginary part* of the complex number. The following notation is sometimes used:

$$Re(z) = a, \quad Im(z) = b$$

In text books you might also see: $\Re(z)$ and $\Im(z)$.

8.1.3 Complex Roots of a Quadratic

We are familiar with the formula for the roots of a quadratic equation:

$$ax^2 + bx + c = 0, \quad x = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$$

Up to now, we would normally say that when $(b^2 - 4ac) < 0$, there are no roots to the equation. But now we discovered (invented!) the way to deal with this situation.

We can now be more precise: when $(b^2 - 4ac) < 0$ there are no *real* roots to the equation, but there will be complex ones!

Example

$$x^2 - 4x + 13 = 0$$

Applying the formula:

$$x = 2 \pm \sqrt{(4-13)} = 2 \pm \sqrt{(-9)} = 2 \pm j3$$

Thus, there are two complex roots to the quadratic equation. This is an important point to note: complex numbers have arisen out of an equation with only real coefficients. By inventing complex numbers we have generalized the applicability of the quadratic roots formula.

In order to test that these really are roots, we would need to test that:

$$(2+j3)(2+j3) - 4(2+j3) + 13 \stackrel{?}{=} 0$$

Obviously, we must develop the rules of arithmetic for complex numbers.

8.2 Algebra of Complex Numbers

Equality

Equality of two complex numbers implies that the real parts are equal, *and* the imaginary parts are equal. So, for

$$z_1 = a + jb, \quad z_2 = c + jd$$

then

$$z_1 = z_2 \quad \Rightarrow a = c, \quad \text{and } \; b = d$$

Addition & Subtraction

The real and imaginary parts are added (or subtracted) separately: For

$$z_1 = a + jb, \quad z_2 = c + jd$$

Then

$$z_1 + z_2 = a + jb + (c + jd) = (a + c) + j(b + d)$$

And

$$z_1 - z_2 = a + jb - (c + jd) = (a - c) + j(b - d)$$

Note that is just like the rule for vector addition using component form.

Multiplication

Multiplication is a straightforward extension of ordinary arithmetic:

$$z_1 \, z_2 = (a + jb)(c + jd) = ac + jcd + jbc + j^2 bd$$

But $j^2 = -1$, so

$$z_1 \, z_2 == ac + jcd + jbc - bd = (ac - bd) + j(ad + bc)$$

Example

We can now look at

$$(2+j3)(2+j3) - 4(2+j3) + 13 = 4 - 9 + j(6+6) - 8 - j12 + 13 = 0$$

Complex Conjugate & Modulus

Take the complex number $(a + jb)$ and multiply by $(a - jb)$:

$$(a + jb)(a - jb) = a^2 - j^2b^2 = a^2 + b^2$$

Note that the result is a *real* number.

The number

$$\bar{z_1} = a - jb$$

is called the **complex conjugate** of z_1. The complex conjugate is formed by reversing the sign on the imaginary part.

Now, (a^2+b^2) is always real and positive, and it is an indication of the "size" (squared) of the complex number. Thus the **modulus** of a complex number is defined as

$$|z_1| = \sqrt{a^2 + b^2} = \sqrt{z_1\, \bar{z_1}}$$

Division

Division of complex numbers seems more difficult:

$$\frac{z_1}{z_2} = \frac{(a + jb)}{(c + jd)}$$

but we can multiply top and bottom by the complex conjugate $\bar{z_2}$ which will make the denominator a real number, which is straightforward to deal with.

$$\frac{z_1}{z_2} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(ac + bd)}{(c^2 + d^2)} + \frac{j(bc - ad)}{(c^2 + d^2)}$$

8.3 Different Representations

In this section we look at the different ways that complex numbers can be written and represented.

8.3.1 The Argand Diagram

The rule for addition of complex numbers is similar to the rule for vector addition in component form, where the x and y components correspond to the real and imaginary parts. This suggests that a complex number $(a + jb)$ can be represented as a *point* (a, b) on a x-y plane. The x axis is called the "real axis" and the y axis is called the "imaginary axis." The diagram which shows this representation of complex numbers is called the **Argand diagram**.

8.3.2 Polar Form

Points in a x-y plane can also be given in terms of polar coordinates (r, θ), such that r is the distance of the point from the origin, and θ is the angle that the line from the origin to the point makes with the x axis. This suggests that there is another way to represent a complex number in polar form.

$$a = r\cos\theta, \quad b = r\sin\theta$$

So that:

$$z = a + jb = r(\cos\theta + j\sin\theta)$$

From Pythagoras:

$$r = \sqrt{a^2 + b^2} = |z|$$

so, r is just the modulus of the complex number.

The angle θ is known as the **argument** of the complex number, written

$$\arg z = \theta$$

However, θ is not unique, since the angles

$$\theta + 2k\pi \quad (k \text{ integer}, \ k \neq 0)$$

are also arguments for the same complex number. The *principal value* for the argument is defined, so as to be single-valued, as

$$-\pi < \theta \leq \pi$$

In order to find r and θ from $z = a + jb$ we use:

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}, \quad \cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

Note, that special care has to be taken to get the correct quadrant for θ.

Sometimes polar form is written using the following notation:

$$z = r\angle\theta$$

Example

$$z = 1 + j\sqrt{3}, \quad \Rightarrow r = \sqrt{1+3} = 2, \quad \sin\theta = \frac{\sqrt{3}}{2}, \quad \cos\theta = \frac{1}{2}$$

Since $\sin\theta$ and $\cos\theta$ are both positive, it must be the case that θ is in the first quadrant (also look at a and b).

$$\theta = \frac{\pi}{3}, \quad \Rightarrow z = 2\angle 60^\circ$$

On the other hand,

$$z = -1 - j\sqrt{3}, \quad \Rightarrow r = \sqrt{1+3} = 2, \quad \sin\theta = \frac{-\sqrt{3}}{2}, \quad \cos\theta = \frac{-1}{2}$$

Since $\sin\theta$ and $\cos\theta$ are both negative, it must be the case that θ is in the third quadrant (but remember the range for the principal value):

$$\theta = -\pi + \frac{\pi}{3}, \quad \Rightarrow z = 2\angle - 120^\circ$$

(It is usually best to draw a sketch of the Argand diagram, in order to get the correct argument value.)

Multiplication

Consider multiplying two numbers in polar form:

$$\begin{aligned} z_1\ z_2 &= [r_1(\cos\theta_1 + j\sin\theta_1)]\,[r_2(\cos\theta_2 + j\sin\theta_2)] \\ &= r_1 r_2\,[\cos\theta_1\ \cos\theta_2 - \sin\theta_1\ \sin\theta_2 + j(\cos\theta_1\ \sin\theta_2 + \sin\theta_1\ \cos\theta_2)] \\ &= r_1 r_2\,[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)] \end{aligned}$$

So, the rule is: *multiply* the moduli, and *add* the arguments.

8.3.3 Exponential Form

Using the rules for multiplication and additions, one can write down a *power series* of a complex variable, i.e., for $z = x + iy$, where x and y are the real and complex parts of the complex variable, so x and y can also be considered as variables:

$$f(z) = a_0 + a_1\,z + a_2\,z^2 + a_3\,z^3 + \ldots$$

It can be shown that the concept of convergence for infinite power series can be generalized to include complex values. We will not pursue this here, but instead just note that the exponential series, in particular, can be shown to converge, so that, for θ real:

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \ldots$$

Manipulating and rearranging

$$\begin{aligned} e^{j\theta} &= 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\right) \end{aligned}$$

So, recognising the bracketed series, we can write

$$e^{j\theta} = \cos\theta + j\sin\theta$$

We can generalize this, since for any complex number:

$$z = a + jb = r(\cos\theta + j\sin\theta)$$

we can now write

$$z = r\,e^{j\theta}$$

This is known as *exponential form.* We have thus found three different representations of the same complex number.

However, and to be more precise, sine and cosine are periodic functions, so there is a more general form:

$$z = r\,e^{j(\theta+2\pi k)}$$

where k is any integer. This follows from, e.g., $\sin(\theta+2\pi k) = \sin\theta$. Shortly we will find out why this form is so important.

Using $(-j)$ instead of (j) in the series expansion gives

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

and adding/subtracting with the first result gives

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) = \cosh(j\theta)$$
$$\sin\theta = \frac{1}{2}\left(e^{j\theta} - e^{-j\theta}\right) = -j\sinh(j\theta)$$

It was an interesting point that it was noted early that the hyperbolic functions obey relations similar to those involving the corresponding circular functions. Here we have found the explicit link between the hyperbolic and circular functions.

Some useful cases

$$\begin{aligned}
\theta = \frac{\pi}{2} \quad &: \quad e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j \\
\theta = \pi \quad &: \quad e^{j\pi} = \cos\pi + j\sin\pi = -1 \\
\theta = \frac{3\pi}{2} \quad &: \quad e^{j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2} = -j \\
\theta = 2\pi \quad &: \quad e^{j2\pi} = \cos 2\pi + j\sin 2\pi = +1
\end{aligned}$$

Generally, for an integer n:

$$\begin{aligned}
e^{jn\pi} &= (-1)^n \\
e^{j2n\pi} &= 1 \\
e^{j(2n+1)\pi/2} &= j(-1)^n
\end{aligned}$$

Multiplication

In exponential form multiplication takes a particular simple form: Suppose $z_1 = r_1 e^{j\theta_1}$, and $z_2 = r_2 e^{j\theta_2}$, then

$$z_3 = z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2\, e^{j(\theta_1+\theta_2)}$$

The rule is: multiply the moduli, and add the arguments.

If $r_2 = 1$ then multiplication by z_2 corresponds to a (anticlockwise) rotation in the Argand diagram of z_1 by an angle θ_2

$$z_3 = z_1\, e^{j\theta_2} = r_1\, e^{j(\theta_1+\theta_2)}$$

Example

Using a complex value for a physical quantity

If we let

$$\theta = \omega t$$

where t is a variable (say time) and ω (omega – the angular frequency) is constant. As t increases, so does the angle argument θ. Then we can consider a *complex function* of t:

$$z(t) = Ae^{j\omega t} = A\left(\cos(\omega t) + j\sin(\omega t)\right)$$

The real part

$$Re(z) = A\cos(\omega t)$$

is a sinusoidal wave as a function of time, and can represent a "real" physical quantity, or signal, (i.e., voltage, current, etc.).

Now consider multiplying by $Be^{j\alpha}$, where B and α are constants.

$$u = Be^{j\alpha}\, z = AB\, e^{j(\omega t+\alpha)}$$

So that the real part becomes

$$Re(u) = AB\cos(\omega t + \alpha)$$

It is evident that the effect of the multiplication has been to shift the *phase* of the signal by an amount α, and its *amplitude* has been multiplied by a factor B. The fact that complex numbers can represent changes in both phase and amplitude means that they are used extensively in signal processing analysis. For example, the effect of a circuit component on a signal can be represented by one complex value, or perhaps by a complex function of the frequency ω (known as the transfer function).

Logarithm of Complex Number

If

$$z = r\, e^{j\theta}$$

then we taking logarithms (assuming that we can do so for a complex value):

$$\ln(z) = \ln r + j\theta$$

Note that, as before, a more general, multi-valued form uses $(\theta + 2\pi k)$ instead of just θ.

--- Example

For

$$z = 1 + j\sqrt{3}$$

the modulus $r = 2$, and $\cos\theta = 1/2$ so $\theta = \pi/3$. So it follows that

$$\ln(1 + j\sqrt{3}) = \ln 2 + j\frac{\pi}{3}$$

8.4 de Moivre's Theorem

Consider

$$z = e^{j\theta} = \cos\theta + j\sin\theta$$

Then powers of z can be written:

$$\begin{aligned} z^2 &= e^{j\theta}e^{j\theta} = e^{j2\theta} = \cos(2\theta) + j\sin(2\theta) \\ z^3 &= e^{j3\theta} = \cos(3\theta) + j\sin(3\theta) \end{aligned}$$

And so on... Generally, we have the result

$$(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta)$$

This result is known as *de Moivre's Theorem*, and can be shown to be true when n is a negative integer, and also when n is a rational fraction such as p/q where p and q are integers.

8.4.1 Roots of a Complex Number

If

$$z^n = re^{j\theta}$$

then, by taking the n-th root and using de Moivre's Theorem

$$z = \sqrt[n]{r}\, e^{j\theta/n} \quad \text{(Not quite right!)}$$

But this cannot be completely correct, since in general there must be n values for the n-th root. The complete result uses the general (multi-valued) form for θ, and then

$$z = \sqrt[n]{r}\, e^{j(\theta+2\pi k)/n}$$

where k is an integer. Typically, one uses $k = 0, 1, 2, 3\ldots$ to find all the different roots.

$$z = \sqrt[n]{r}\, e^{j\theta/n}, \quad \sqrt[n]{r}\, e^{j\theta/n}e^{j2\pi/n}, \quad \sqrt[n]{r}\, e^{j\theta/n}e^{j4\pi/n}, \quad \ldots$$

At some stage the values for the roots will begin to repeat themselves.

Example

Suppose

$$z^2 = 4e^{j\pi/3}$$

which can be written in multi-valued form as:

$$z^2 = 4e^{j(\pi/3+2\pi k)}$$

The roots are then

$$z = 2\, e^{j\pi/6}, \quad 2\, e^{j(\pi/6+\pi)}, \quad 2\, e^{j(\pi/6+2\pi)}, \ldots$$

Note that for $k = 0$ and $k = 1$ the values are distinct, but the value for $k = 2$ is the the same as that for $k = 0$, and so on. Thus there are only two distinct roots.

8.4.2 The Roots of Unity

We can now find the n-th roots of unity, by noting that

$$1 = e^{j2\pi k}, \quad k = 0, 1, 2, 3, \ldots$$

For example, the cube roots of unity are

$$\sqrt[3]{1} = e^{j2\pi k/3}, \quad k = 0, 1, 2$$

So,

$$\begin{aligned}\sqrt[3]{1} &= 1, \quad \left[\cos\left(\frac{2\pi}{3}\right) + j\sin\left(\frac{2\pi}{3}\right)\right], \quad \left[\cos\left(\frac{4\pi}{3}\right) + j\sin\left(\frac{4\pi}{3}\right)\right] \\ &= 1, \quad \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right), \quad \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\end{aligned}$$

Note, again, that the values for $k > 2$ just repeat themselves. When these values are plotted on the Argand diagram, it is apparent how roots are related to repeated rotations in the complex plane.

8.4.3 Expansions for $\sin^n\theta$, $\sin(n\theta)$, etc.

Let

$$z = e^{j\theta} = \cos\theta + j\sin\theta$$

We have that

$$z^n = \cos(n\theta) + j\sin(n\theta)$$

and

$$z^{-n} = \frac{1}{z^n} = \cos(n\theta) - j\sin(n\theta)$$

Adding/subtracting, and rearranging...

$$z^n + \frac{1}{z^n} = 2\cos(n\theta)$$
$$z^n - \frac{1}{z^n} = 2j\sin(n\theta)$$

And in particular:

$$z + \frac{1}{z} = 2\cos\theta, \quad z - \frac{1}{z} = 2j\sin\theta$$

Using these identities expansions for powers of sine and cosine can be found.

For example,

$$\begin{aligned}
(2\cos\theta)^3 = 8\cos^3\theta &= \left(z + \frac{1}{z}\right)^3 \\
&= \left(z + \frac{1}{z}\right)\left(z^2 + 2 + \frac{1}{z^2}\right) \\
&= z^3 + \frac{1}{z^3} + 2\left(z + \frac{1}{z}\right) + \left(\frac{z}{z^2} + \frac{z^2}{z}\right) \\
&= \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right) \\
&= 2\cos(3\theta) + 6\cos\theta
\end{aligned}$$

The process can also be reveresed to find expansions for $\cos(n\theta)$ and $\sin(n\theta)$ in terms of powers of sine and cosine. For example:

$$\begin{aligned}
\cos(4\theta) + j\sin(4\theta) &= (\cos\theta + j\sin\theta)^4 \\
&= \cos^4\theta + j4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - j4\cos\theta\sin^3\theta + \sin^4\theta
\end{aligned}$$

Equating the real and imaginary parts gives:

$$\begin{aligned}
\cos(4\theta) &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\
\sin(4\theta) &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta
\end{aligned}$$

Example

Alternative example:

$$\begin{aligned}
\cos(3\theta) + j\sin(3\theta) &= (\cos\theta + j\sin\theta)^3 \\
&= \cos^3\theta + 3\cos^2\theta(j\sin\theta) + 3\cos\theta(j\sin\theta)^2 + (j\sin\theta)^3 \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta + j(3\cos^2\theta\sin\theta - \sin^3\theta)
\end{aligned}$$

8.5 Loci Problems

Inequalities that involve complex numbers have an interpretation in terms of lines, points and areas in the Argand diagram. Thus it is possible to specify points or regions (i.e., loci – a set of points) in terms of such inequalities.

Working out the region or points corresponding to a certain complex inequality involves: common sense, reduction to cartesian form via complex algebra, and a little cunning.

Example

Find region in complex plane (Argand diagram) that satisfies

$$0 < \arg(z) < \frac{\pi}{2}$$

Here the modulus doesn't matter, only the angle argument, so the above inequality defines the first quadrant of the complex plane.

Example

Find the set of points satisfying

$$|z| = 2$$

Here the modulus is fixed (i.e., distance from origin), but the angle argument is arbitrary. Thus the set of points is a circle of radius 2, centred on the origin.

On the other hand:

$$\arg(z) = \frac{\pi}{4}$$

defines a straight line, on which all points have the given angle argument.

Example

Locus satisfying

$$|z - 2| = 3$$

The left hand side represents the distance from the point $(2, 0)$ (i.e., 2+j0). This follows from a vector-like interpretation of the Argand diagram. Together with the right hand side the inequality says that this distance is fixed and equal to 3. Thus the locus describes a circle of radius 3, centred on the point $(2 + j0)$.

It follows that

$$|z - 2| < 3$$

describes the *interior* of the circle, and

$$|z - 2| > 3$$

describes the *exterior* of the circle.

The same result can be found algebraically. Using

$$z = x + jy$$

Then

$$|z - 2| = |x - 2 + jy| = \sqrt{(x - 2)^2 + y^2)} = 3$$

So

$$(x - 2)^2 + y^2 = 9$$

which describes a circle of radius 3, centred on the point $(2, 0)$.

Example

Find locus of points satisfying

$$|z - 1| = \frac{1}{2}|z - j|$$

Use the substitution

$$z = x + jy$$

the condition then becomes

$$\begin{aligned} |(x-1)+jy| &= \frac{1}{2}|x+j(y-1)| \\ 4(x-1)^2+4y^2 &= \left(x^2+(y-1)^2\right) \\ 4x^2-8x+4+4y^2-x^2-y^2-1+2y &= 0 \\ 3x^2-8x+3y^2+2y+3 &= 0 \end{aligned}$$

This is the sum of two quadratics in x and y, and so can one can complete the square on each separately

$$\begin{aligned} 3\left(x^2-\frac{8}{3}x\right) &= 3\left(x-\frac{4}{3}\right)^2-\frac{16}{3} \\ 3\left(y^2-\frac{2}{3}y\right) &= 3\left(y-\frac{1}{3}\right)^2-\frac{1}{3} \end{aligned}$$

Substituting back into the relation

$$3\left(x-\frac{4}{3}\right)^2+3\left(y-\frac{1}{3}\right)^2=\frac{8}{3}$$

Or,

$$\left(x-\frac{4}{3}\right)^2+\left(y-\frac{1}{3}\right)^2=\frac{8}{9}$$

This is the equation of a circle with radius $\sqrt{8}/3$ centred at the point

$$\left(\frac{4}{3},\frac{-1}{3}\right)$$