ON THE ORDERS OF AUTOMORPHISM GROUPS OF FINITE GROUPS. II

JOHN N. BRAY AND ROBERT A. WILSON

Abstract

In the Kourovka Notebook, Deaconescu asks if $|\text{Aut } G| \geq \phi(|G|)$ for all finite groups $G$, where $\phi$ denotes the Euler totient function; and whether $G$ is cyclic whenever $|\text{Aut } G| = \phi(|G|)$. In an earlier paper we have answered both questions in the negative, and shown that $|\text{Aut } G|/\phi(|G|)$ can be made arbitrarily small. Here we show that these results remain true if $G$ is restricted to being perfect, or soluble.

1. The question, and general overview

Let $\phi$ denote the Euler totient function, so that $\phi(n)$ is the number of integers $m$ with $1 \leq m \leq n$ such that $m$ and $n$ are coprime, and

$$\frac{\phi(n)}{n} = \prod_{i=1}^{r} \frac{p_i - 1}{p_i},$$

where $p_1 < p_2 < \ldots < p_r$ are the prime factors of $n$. It is easy to see that for finite abelian groups $G$, we have $|\text{Aut } G| \geq \phi(|G|)$, with equality if and only if $G$ is cyclic.

In [1] we showed that neither statement holds for arbitrary finite groups, thus solving Problem 15.43 of the Kourovka Notebook [5].

On the other hand they hold (trivially) for finite simple groups (indeed for all finite groups with trivial centre), and one is led to ask: For what classes of finite groups do the statements hold?

A long-standing conjecture of Schenkman [6], that if $G$ is a finite non-cyclic $p$-group of order at least $p^3$ then $|G|$ divides $|\text{Aut } G|$, would imply that both statements hold for finite nilpotent groups. Indeed, this is known to hold for nilpotent groups of class 2, see Schenkman [6].

In this paper we show that the statements do not hold for the class of perfect groups, nor for the class of soluble groups. As in [1], we actually prove stronger results:

**Theorem 1.** For all $\varepsilon > 0$ there exists a finite perfect group $G$ such that $|\text{Aut } G| < \varepsilon \cdot \phi(|G|)$.

**Theorem 2.** For all $\varepsilon > 0$ there exists a finite soluble group $G$ such that $|\text{Aut } G| < \varepsilon \cdot \phi(|G|)$.

**Theorem 3.** For all $N \in \mathbb{N}$ there exists a finite perfect group $G$ with $|G| > N$ such that $|\text{Aut } G| = \phi(|G|)$.

2000 Mathematics Subject Classification 20E36 (primary), 20F28 (secondary).
Theorem 4. For all \( N \in \mathbb{N} \) there exists a finite non-cyclic soluble group \( G \) with \( |G| > N \) such that \( |\text{Aut } G| = \phi(|G|) \).

We were unable to resolve the case of supersoluble groups, but are marginally inclined to the view that:

Conjecture. If \( G \) is a finite non-nilpotent supersoluble group, then \( |\text{Aut } G| > \phi(|G|) \).

Conventions. Throughout this paper, we shall only consider finite groups. The notation for group structures is based on that used in the Atlas [3]. The notation \( O_p(G), O'_p(G), O_p^t(G), \text{Aut } G, \) and \( \text{Out}(G) \) is standard. The abbreviation PIM stands for projective indecomposable module. If \( U \) and \( V \) are modules then \( U \cdot V \) denotes a non-split extension of \( U \) by \( V \) with \( U \) being the submodule and \( V \) being the quotient.

2. Some modules and cohomology

We need some information about modules and cohomology of \( L_2(p) \), especially when \( p \equiv 7 \pmod{8} \). The following information was established in [1]:

Lemma 5. For \( p \) prime and \( p \equiv 7 \pmod{8} \) there are precisely two isomorphism classes of \( \mathbb{F}_p L_2(p) \)-modules \( 1 \cdot U \) in which \( U \) is absolutely irreducible of dimension \( \frac{1}{2}(p-1) \), and the \( 1 \) denotes the trivial module. These two modules are interchanged by the non-trivial outer automorphism of \( L_2(p) \), and both of these modules have zero 1-cohomology. These two modules have the forms \( 1 \cdot U_1 \) and \( 1 \cdot U_2 \) where \( U_1 \) and \( U_2 \) are not isomorphic.

For all primes \( p \) there are just \( p \) irreducible modules of \( \text{SL}_2(p) \) in characteristic \( p \). Their dimensions are all different, and at most \( p \), and we label the \( \text{SL}_2(p) \)-irreducible of dimension \( i \) \((1 \leq i \leq p)\) as \( V_i \). For \( p \) odd, the central involution of \( \text{SL}_2(p) \) acts trivially on \( V_i \) if and only if \( i \) is odd; in such cases we regard \( V_i \) as being an \( L_2(p) \) module. Of course, \( V_1 \) is the trivial module for \( L_2(p) \).

For \( p \equiv 3 \pmod{4} \) the Brauer tree of the principal block of \( L_2(p) \) in characteristic \( p \) is a straight line with \( \frac{1}{2}(p+1) \) nodes and diagram

\[
\begin{align*}
V_1 & \quad V_{p-2} & \quad V_3 & \quad \ldots \ldots & \quad V_{\frac{1}{2}(p-1)} \\
1 & \quad p-1 & \quad p+1 & \quad p-1 & \quad p+1 & \quad \frac{1}{2}(p-1)
\end{align*}
\]

where we have labelled the nodes with the degrees of ordinary characters to which they correspond and we have labelled the edges with their corresponding \( p \)-modular irreducibles. From the Brauer tree one reads off the PIMs

\[ V_1 \cdot V_{p-2} \cdot V_1 \quad \text{and} \quad V_{p-2} \cdot (V_1 \oplus V_3) \cdot V_{p-2} \]

for all primes \( p \geq 7 \) with \( p \equiv 3 \pmod{4} \). (In fact, these PIM structures are valid for all primes \( p \geq 5 \).) Note that the \( V_i \) and all of the PIMs for \( L_2(p) \) (and also \( \text{SL}_2(p) \)) can be realised over \( \mathbb{F}_p \).

Let \( W \) be the \( \mathbb{F}_p L_2(p) \)-module \( (V_1 \oplus V_3) \cdot V_{p-2} \) (with simple head). So \( W \) is a
ON THE ORDERS OF AUTOMORPHISM GROUPS OF FINITE GROUPS. II

quotient of the PIM $V_{p-2} \cdot (V_3 \oplus V_4) \cdot V_{p-2}$ and therefore is unique. One can also read off from the PIMs that $W$ has zero 1-cohomology whenever $p \geq 7$.

For $p$ prime and $p \equiv 7 \pmod{8}$, we define $J_p$ to be $J_p \cong (2^{(p+1)/2} \times p^{p+2}) : L_2(p)$, in which the complementary $L_2(p)$ act on $O_p(J_p)$ as the module $W \cong (1 \oplus V_3) \cdot V_{p-2}$ and on $O_2(J_p)$ as the module $1 \cdot U_1$ of Lemma 5. The groups $J_p/O_p(J_p)$ are isomorphic to the groups $M_p$ we constructed in [1].

3. Perfect groups

In this section, we construct infinite series of finite perfect groups which prove Theorems 1 and 3. We let $r \geq 11$ be a prime, and define $G$ to be the direct product of certain perfect groups $B_p$ for each prime $p$ between 3 and $r$ inclusive:

$$G = \prod_{p \in \pi} B_p = \prod_{p=3, \text{p prime}}^r B_p,$$

where $\pi$ is the set of odd primes not exceeding $r$. Firstly, we take $B_3 \cong 3^6 : M_{11}$, where $O_3(B_3)$ when regarded as an $F_3M_{11}$-module is a uniserial module of shape $1 \cdot 5a$ (this module is isomorphic to the unique 6-dimensional submodule of the $F_3$-permutation module of $M_{11}$ on the 12 cosets of $L_2(11)$). Note also that the composition factor $5a$ is absolutely irreducible. We have:

**Lemma 6.** The $F_3M_{11}$-module $1 \cdot 5a$ has zero 1-cohomology.

**Proof.** This is an easy calculation using Magma [2]. Alternatively, the trivial module has zero 1-cohomology since $M_{11}$ is perfect, and it can be shown that the module $5a$ does not occur in the second Loewy layer of the trivial PIM. Thus both composition factors of $1 \cdot 5a$ have trivial 1-cohomology, and so does the whole module.

**Lemma 7.** If $p = 3$, so that $B_p = B_3 \cong 3^6 : M_{11}$, then $B_p = B_3$ has outer automorphism group of order 2. Thus $|\text{Aut } B_p| = \frac{2}{3} |B_p| = \frac{p-1}{p} |B_p|$.

**Proof.** Since $O_3(B_3)$ is a characteristic subgroup of $B_3$, any automorphism of $B_3$ permutes the complements to $O_3(B_3)$ in $B_3$. Now let $S$ denote a complementary $M_{11}$ in $B_3$. Since we have ensured that the $F_3M_{11}$-module $O_3(B_3)$ has zero 1-cohomology, we may assume our automorphism, $\alpha$ say, normalises $S$. But $M_{11}$ has trivial outer automorphism group, and adjusting $\alpha$ by an inner automorphism that is conjugation by an element of $S$, we may assume that $\alpha$ centralises $S$. So now $\alpha$ is an $F_3S$-module automorphism of $O_3(B_3) \cong 1 \cdot 5a$, and is thus a non-zero scalar. There are two of these and so $|\text{Out } B_p| = 2$. In fact, $\text{Aut } B_p \cong 3^2 : (M_{11} \times 2)$.

For $p \geq 5$ and $p \equiv 7 \pmod{8}$, we take $B_p \cong p^{1+2} : SL_2(p)$. For $p \geq 5$ and $p \equiv 7 \pmod{8}$, we take $B_p \cong p^{1+2} : SL_2(p)$ or $B_p \cong J_p \cong (2^{(p+1)/2} \times p^{p+2}) : L_2(p)$, the group we constructed in Section 2 (we are free to choose either; this choice is necessary in order to prove Theorem 3).

The group $p^{1+2} : SL_2(p)$ has a centre of order $p$, and is isomorphic to a vector stabiliser in the natural representation of $Sp_4(p)$.
LEMMA 8. Let $H = N:K$ and let $K$ act faithfully on $N$. Suppose $\alpha \in \operatorname{Aut} H$ centralises $N$ and normalises $K$. Then $\alpha$ centralises $K$. (So $\alpha = 1$.)

Proof. For all $g \in N$, $k \in K$ we have $(g^k)\alpha = g^k$ since $g^k \in N$. On the other hand $(g^k)\alpha = (g\alpha)^{(k\alpha)} = g^{(k\alpha)}$. So for all $g \in N$, $k \in K$ we have $g^{(k\alpha)^{-1}} = g$, whence $k\alpha = k$ since $K$ acts faithfully on $N$.

LEMMA 9. For all primes $p \geq 5$ we have $\operatorname{Aut}(p^{1+2}:\operatorname{SL}_2(p)) \cong p^2:GL_2(p)$. So if $B_p \cong p^{1+2}:\operatorname{SL}_2(p)$ we have $|\operatorname{Aut} B_p| = \frac{p-1}{p} |B_p|$.

Proof. The 1-space stabiliser in $\operatorname{Sp}_2(p)$ is a group $p^{1+2}:\operatorname{GL}_2(p)$ which induces a group $p^2:GL_2(p)$ of automorphisms on its normal subgroup $p^{1+2}:\operatorname{SL}_2(p)$.

The group $p^{1+2}:\operatorname{SL}_2(p)$ contains exactly $p^2$ involutions, which are permuted faithfully by the above group $p^2:GL_2(p)$. Each of these involutions has centraliser of shape $p \times \operatorname{SL}_2(p)$, and these define the $p^2$ complements $\operatorname{SL}_2(p)$ [by taking the $O^p$ or the derived subgroup]. Moreover, these involutions generate $p^{1+2}:2$, and support a natural affine plane structure; three involutions are collinear in this affine plane if and only if they generate a subgroup isomorphic to $D_{2p}$.

We already see the full automorphism group $p^2:GL_2(p)$ of this affine plane, so the only way the automorphism group of $p^{1+2}:\operatorname{SL}_2(p)$ could be any bigger is if there were a non-trivial kernel, i.e. an automorphism centralising all $p^2$ involutions. Such an automorphism would have to normalise, and therefore by Lemma 8 centralise, each of the complements, as well as centralising the group $p^{1+2}:2$ generated by the involutions. Therefore it is the trivial automorphism on each complementary $\operatorname{SL}_2(p)$, and hence on $p^{1+2}:\operatorname{SL}_2(p)$, and the lemma follows.

LEMMA 10. If $p \equiv 7 \pmod{8}$ and $H$ is the group $J_p \cong (2^{p+1}/2 \times p^{p+2}):L_2(p)$ we constructed in Section 2, then $\operatorname{Aut} H \cong (2^{p-1}/2 \times p^{p+1}):\langle L_2(p) \times C_p \rangle$. So if $B_p \cong J_p$ we have $|\operatorname{Aut} B_p| = \frac{p-1}{p^2} |B_p|$.

Proof. The elementary abelian subgroups $O_2(H)$ and $O_p(H)$ are characteristic in $H$; therefore $K := O_2(H) \times O_p(H)$ is also characteristic in $H$. Since $O_2(H)$ and $O_p(H)$ both have zero 1-cohomology as $L_2(p)$-modules (see Section 2), $H$ has just one conjugacy class of complementary subgroups $L_2(p)$. So let $\alpha \in \operatorname{Aut} H$ and let $S$ be a complementary subgroup $L_2(p)$. Modulo inner automorphisms, $\alpha$ normalises $S$. Now $O_2(H)$ when regarded as an $F_2S$-module does not admit the non-trivial outer automorphism of $S \cong L_2(p)$, see Lemma 5. So $\alpha$ induces an inner automorphism when restricted to $S$, and adjusting by an inner automorphism of $H$ that is conjugation by an element of $S$, we may assume that $\alpha$ centralises $S$. So now $\alpha$ induces an $F_2S$-module automorphism on $O_2(H)$ and an $F_pS$-module automorphism on $O_p(H)$, and both of these are scalars. Since $H \cong J_p$ has centre of order $2p$, the result follows.

LEMMA 11. For all primes $p$ with $3 \leq p \leq r$, the groups $B_p$ are characteristic in $G$.

Proof. Let $\pi$ be the set of all primes between 3 and $r$ inclusive. Let $N = F(G)$,
the Fitting subgroup of $G$, so that $N$ is characteristic in $G$. Then
\[ G/N \cong \prod_{p \in \pi} S_p, \]
where $S_3 \cong M_{11}$ and $S_p \cong L_2(p)$ whenever $p \geq 5$. So $G/N$ has a unique normal subgroup $N_p/N$ such that $N_p/N \cong S_p$, and for all $p$ we get that $N_p$ is characteristic in $G$. In fact
\[ N_p = B_p \times \prod_{q \in \pi'} O_{(2,q)}(B_q), \]
where $\pi' = \pi \setminus \{p\}$, with the $O_{(2,q)}(B_q)$ being nilpotent groups of class at most 2. Therefore $B_p \cong N''_p$ is a characteristic subgroup of $G$.

Since all of the $B_p$ are characteristic in $G$, we have $\text{Aut} G \cong \prod_{p \in \pi} \text{Aut} B_p$. We have also established for all $p \in \pi$ that $|\text{Aut} B_p| = \frac{p-1}{p^2} |B_p|$ or $\frac{1}{2} \frac{p-1}{p^2} |B_p|$, with the latter case occurring if and only if $B_p \cong J_p$. Therefore we have
\[ |\text{Aut} G| \cong \frac{1}{2m-1} \times \prod_{p \in \pi} \frac{p-1}{p} \times |G| = \frac{1}{2m-1} \times \phi(|G|). \]

When $m = 1$ this gives $|\text{Aut} G| = \phi(|G|)$. We now invoke Dirichlet’s Theorem that there are infinitely many primes $p$ with $p \equiv 7 \pmod{8}$ to complete the proofs of Theorems 1 and 3.

**Remark.** It is convenient but not essential to take all odd primes up to $r$ in the definition of $G$. But every odd prime dividing $|G|$ must be one of these defining primes. To this end, let $\pi$ be a set of odd primes such that $3, 5, 11 \in \pi$ and if $p \in \pi$ then $q \in \pi$ whenever $q$ is an odd prime factor of $p - 1$ or $p + 1$. Let the $B_p$ be as above. Then the group
\[ G = \prod_{p \in \pi} B_p \]
satisfies $|\text{Aut} G| = 2^{1-m} \phi(|G|)$ where $m$ is the number of $p \in \pi$ such that $B_p \cong (2^{(p+1)/2} \times p^{p+2}) \cdot L_2(p)$.

4. **Soluble groups**

In this section, we construct infinite series of finite soluble groups in order to prove Theorems 2 and 4.

We define $B_3$ to be the unique group of shape $3_1^{1+2}:4$ with centre of order 3; this group is $\text{SmallGroup}(108,15)$ in various versions of Magma [2], including Version 2.10. Let $\pi$ be a finite non-empty set of primes such that $p \equiv 1$ or $7 \pmod{8}$ for all primes $p > 3$. Then
\[ |\text{Aut} G| = 2^{1-m} \phi(|G|) \]
where $m$ is the number of $p \in \pi$ such that $B_p \cong (2^{(p+1)/2} \times p^{p+2}) \cdot L_2(p)$.
Now $B$ of order 4. Since $H$ is a characteristic subgroup of $G$, we calculate that $p$ in $SL_K$ the corresponding subgroup $H$ generated by the elements of order 4. Since $H_G$ is a characteristic subgroup of $\alpha$. Therefore $z$ of $G$ is also.

Lemma 12. For all $p \in \pi$, the groups $B_p$ are characteristic in $G$.

Proof. Let $\pi' := (\pi \cup \{3\}) \setminus \{p\}$. The group $O_2^+(G)$ is the direct product of the subgroups $W_p := O_p(B_p) \cong p_+^{1+2}$. We have

$$O_{(2,p)}^+(G) = O_p(O_2^+(G)) = \prod_{q \in \pi'} W_q.$$ 

Thus we calculate that

$$H := C_G(O_p(O_2^+(G))) = B_p \times \prod_{q \in \pi'} Z(W_q) \cong p_+^{1+2} \times \prod_{q \in \pi'} C_q.$$ 

Now $B_p$ is characteristic in $H$, since it is the subgroup generated by the elements of order 4. Since $H$ is a characteristic subgroup of $G$ we conclude that $B_p$ is also.

Lemma 13. The group $B_3$ is also characteristic in $G$.

Proof. This proof is very similar to the proof of Lemma 12, and we use the notation $W_q$ from the proof of that lemma here. The group

$$H := C_G(O_{2}^+(O_2^+(G))) = B_3 \times \prod_{q \in \pi} Z(W_q) \cong 3_4^{1+2} \times \prod_{q \in \pi} C_q$$

is a characteristic subgroup of $G$. Now $B_p$ is characteristic in $H$, since it is the subgroup generated by the elements of order 4. Since $H$ is a characteristic subgroup of $G$ we conclude that $B_p$ is also.

Lemma 14. For all $p \in \pi$, we have $\text{Aut} B_p \cong p_2^2 : (2S_4^- \circ C_{p-1})$. So for all such primes $p$ we have $|\text{Aut} B_p| = \frac{3(p-1)}{2} p^2 |B_p|$.

Proof. We adapt the proof of Lemma 9. First note that $H = B_p \cong p_+^{1+2} : 2S_4^-$ embeds in $p_+^{1+2} : \text{GL}_2(p)$, in which its normaliser is $p_+^{1+2} : (2S_4^- \circ C_{p-1})$, and therefore its automorphism group contains $p_2^2 : (2S_4^- \circ C_{p-1})$.

Note that the latter group acts transitively and faithfully on the $p^2$ involutions in $H$, so if we have any further automorphism $\alpha$, we may assume $\alpha$ fixes one of these involutions, say $z$. Therefore $\alpha$ fixes $C_H(z) \cong p \times 2S_4^-$, and therefore normalises the corresponding subgroup $K = O_2^+(C_H(z)) \cong 2S_4^-$ of $H$.

Now both $\alpha$ and $K$ act on the affine plane defined by the $p^2$ involutions (as in
Lemma 9), and both fix the same point, which we can regard as the origin. In the resulting action on the vector space of order \( p^2 \), the image of \( \alpha \) normalises the image of \( K \) inside \( \text{GL}_2(p) \) (indeed, \( K: \langle \alpha \rangle \) acts on this vector space). But this action of \( K \) is faithful and \( N_{\text{GL}_2(p)}(2S_4^{\pm}) \cong 2S_4^{\pm} \circ C_{p-1} \). But we have already seen a group \( p^2:(2S_4^{\pm} \circ C_{p-1}) \) of automorphisms of \( H \) acting faithfully on the affine plane, and so we may assume that \( \alpha \) acts trivially on the affine plane. In other words, \( \alpha \) centralises all \( p^2 \) involutions in \( H \), so centralises the group \( p^1+2:2 \) which they generate.

We now know that \( \alpha \) centralises \( p^1+2 \), and normalises a complementary \( 2 \cdot \sigma - 4 \).

Therefore, by Lemma 8, \( \alpha \) is the trivial automorphism of \( H \). This completes the proof of the lemma.

An easy calculation gives \( \text{Aut} B_3 \cong 3^2:SD_{16} \). Since all of the \( B_p \) are characteristic in \( G \), we obtain \( \text{Aut} G \cong \prod_{p \in \pi \cup \{3\}} \text{Aut} B_p \). Therefore we have

\[
\frac{|\text{Aut} G|}{|G|} = \frac{4}{3} \times \left( \frac{1}{2} \right)^{|\pi|} \prod_{p \in \pi} \frac{p-1}{p} = \frac{1}{2^{|\pi|-2}} \times \frac{\phi(|G|)}{|G|},
\]

and so

\[
\frac{|\text{Aut} G|}{\phi(|G|)} = \frac{1}{2^{|\pi|-2}}.
\]

When \(|\pi| = 2\) this gives \( |\text{Aut} G| = \phi(|G|) \). Dirichlet’s Theorem tells us that \( \pi \) can be made arbitrarily large, thus proving Theorem 2, and also that there are infinitely many size 2 possibilities for \( \pi \), thus proving Theorem 4.

**Remark.** In the above construction for \( G \) we can replace the group \( B_3 \) by the cyclic group of order 3, in which case \( |\text{Aut} G| = 2^{1-|\pi|} \phi(|G|) \). However the proofs are slightly different. The smallest non-cyclic group \( G \) we know of that satisfies \( |\text{Aut} G| = \phi(|G|) \) is now the group \( G \cong 3 \times 7^1+2:2 S_4^{\pm} \) of order 49392, narrowly beating the example \( 2^3:L_3(2) \times 3 \times 7 \) of order 56448 that we gave in [1].

**Acknowledgements.** We are grateful to Chris Parker for helping us simplify some of the proofs in this paper.

**References**

John N. Bray and Robert A. Wilson.
School of Mathematics and Statistics,
University of Birmingham,
Edgbaston, Birmingham, B15 2TT.

jnb@maths.bham.ac.uk
R.A.Wilson@bham.ac.uk