A new family of modules with 2-dimensional 1-cohomology

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Abstract

It is known that for finite simple groups it is possible for a faithful absolutely irreducible module to have 1-cohomology of dimension at least 3. However, even faithful absolutely irreducible modules with 2-dimensional 1-cohomology are rare. We exhibit a new infinite family of such modules.

1 Introduction

Over the past few decades, a number of first cohomology groups have been computed, see for example [1, 2, 8, 5, 6]. The first cohomology group $H^1(G; M)$ of a group $G$ at a module $M$ has two fundamental interpretations: first, it parametrises the complements of $M$ in the split extension $M:G$ of $M$ by $G$; and second, it parametrises the extensions of $M$ by the trivial $G$-module. In what follows we assume that $M$ is a faithful absolutely irreducible module for $G$, since $\dim H^1(G; M)$ can be made arbitrarily large if either of these conditions fails.

In most cases which have been calculated up to now, $H^1(G; M)$ has dimension 0 or 1, and in the few remaining cases it has dimension 2 or 3. The known examples of 3-dimensional 1-cohomology are given in Bray and Wilson [4] and Scott [9]. The two examples of [4] are explicit, whilst despite the infinitude of examples in [9], the results therein do not allow one to give an explicit example of a module with 3-dimensional 1-cohomology.
The results of this paper were discovered when I was carrying out a systematic calculation of 1-cohomologies (and 2-cohomologies) of (absolutely) irreducible modules of simple (and almost simple) groups, in connection with the Web-Atlas project [10]. Many of the 2-dimensional 1-cohomologies seen have an exceptional feel to them, such as $\dim H^1(A_6; F_4^3) = 2$ (the smallest such example). During the course of this search I found that $\dim H^1(L_3(7); F_5^{19}) = 2$, while for other irreducible $L_3(7)$-modules the 1-cohomology is at most 1-dimensional. This looked odd, especially when you compare the complexity of the Sylow 3-subgroup ($3^2$) with that of the Sylow 7-subgroup ($7^{1+2}$), and led to the Main Theorem. The smallest example which is part of this series, namely that $\dim H^1(L_3(4); F_3^{19}) = 2$, was discovered earlier but somehow did not raise any eyebrows.

**Main Theorem.** Let $n \geq 3$, let $p$ be a prime such that $p \mid n$, let $q$ be a prime power such that $q \equiv 1 \pmod{p}$, and let $k$ be a field of characteristic $p$. Then $\dim kH^1(L_n(q); M) = 2$ where $M$ is a certain absolutely irreducible $kL_n(q)$-module of dimension $\frac{q^n - 1}{q - 1} - 2$. We obtain $M$ as the non-trivial composition factor of the permutation module of degree $\frac{q^n - 1}{q - 1}$ for $L_n(q)$ over $k$, corresponding to the action of $SL_n(q)$ on the projective points (or hyperplanes) of the natural $n$-dimensional $F_qSL_n(q)$-module.

In addition to the above family of cross characteristic examples, the following examples of 2-dimensional 1-cohomology in defining characteristic are also known. Cline, Jones, Parshall and Scott [8, 5] give the examples of $\Omega^+_{4m}(q)$ for $q$ even, $q > 2$ and $m \geq 2$ acting on a module $M$ of dimension $2m(4m - 1) - 2$, where $M$ is the non-trivial composition factor of $\Lambda^2(V)$, with $V$ being a natural module of $\Omega^+_{4m}(q)$. They also point out that the ‘corresponding’ (i.e. 26-dimensional) module of $3D_4(q)$ for $q$ even, $q > 2$ also has 2-dimensional 1-cohomology. The above results probably hold when $q = 2$. For the groups $\Omega^+_{4m}(2), m \geq 2$ see [8, 5], while we have done explicit computations for the relevant modules of $\Omega^-_{4m}(2)$ for $4m \in \{8, 12, 16, 20, 24, 28\}$ and $3D_4(2)$. In all cases, a subquotient of the tensor square of the natural module exhibits the 2-dimensional 1-cohomology.

Other than the above families of examples we know explicitly of only finitely many faithful absolutely irreducible modules for which the 1-cohomology is at least 2-dimensional.

## 2 Proof of the Main Theorem

Firstly, we establish some notation. The ATLAS [7] notation is used for group structures, with $E_q$ denoting an elementary abelian group of order $q$. For $G$ a group and $k$ a field, we also use $k$ to denote the trivial $kG$-module. If $V$, $U$ and $W$ are modules, we write $V \cong U \cdot W$ to mean that $V$ has a submodule (isomorphic to) $U$ and $V/U \cong W$; and we write $V \cong U \cdot W$ if $V \cong U \cdot W$ and $U$ does not have a complement isomorphic to $W$. The standard results we quote here can be found, for example, in Benson’s book [3] (which states them in more generality).

Our example involves a certain module of $L_n(q)$ in characteristic $p$, where $n \geq 3$, $p \mid n$ and $q \equiv 1 \pmod{p}$. We first recall some well-known facts about the groups $L_n(q) = PSL_n(q)$.
and \( SL_n(q) \). Firstly, for \( n \geq 2 \) and \( (n, q) \neq (2, 2) \) or \( (2, 3) \) the groups \( L_n(q) \) are simple and the groups \( SL_n(q) \) are perfect and quasi-simple. Secondly, the Schur multiplier of \( SL_n(q) \) is always trivial, except for the cases \( (n, q) = (2, 4), (2, 9), (3, 2), (3, 4) \) or \( (4, 2) \), and the \( r' \)-part of the Schur multiplier of \( SL_n(q) \) is always trivial where \( r \) is the prime such that \( r \mid q \).

It is known that the stabiliser of a projective point (i.e. 1-space) in \( SL_n(q) \) is a subgroup \( H \) of shape \( E_q^{n-1}.GL_{n-1}(q) \), which contains a subgroup \( E_q^{n-1}.SL_{n-1}(q) \cong ASL_{n-1}(q) \) to index \( q - 1 \). We note that \( H \) need not be isomorphic to \( AGL_{n-1}(q) \); in particular, we have non-isomorphism in the cases of interest.\(^1\) Since \( n \) and \( n - 1 \) are coprime, the subgroup of scalars of \( SL_n(q) \) intersects the said subgroup \( ASL_{n-1}(q) \) trivially, and thus the image of this subgroup in any of the images \( C.L_n(q) \) of \( SL_n(q) \) is still isomorphic to \( ASL_{n-1}(q) \). Thus any non-trivial representation of \( SL_n(q) \) restricts faithfully to the subgroup \( ASL_{n-1}(q) \).

So now let \( n \geq 3, p \mid n, p \mid (q - 1) \), \( G \cong SL_n(q) \), \( H \cong E_q^{n-1}.GL_{n-1}(q) \) (the point stabiliser), and let \( k \) be a field of characteristic \( p \). Let \( P \) be the permutation module over \( k \) of the cosets of \( H \). Thus \( P \) is obtained by inducing the trivial \( kH \)-module up to \( G \). Since \( Z(G) \leq H \), \( Z(G) \) acts trivially on \( P \), and thus \( P \) is a (permutation) module for \( L_n(q) \). The dimension of \( P \) is \( m := q^{n-1}.q - 1 \). Permutation modules for transitive groups have a unique trivial submodule, \( U \) say, generated by the all 1s vector \( u = \sum_{i=1}^{m} e_i \); and a unique trivial quotient, whose kernel is the augmentation submodule, \( W \) say, where \( W = \{ \sum_{i=1}^{m} a_i e_i : a_i \in k \mid \sum_{i=1}^{m} a_i = 0 \} \). We have \( U \leq W \) since \( p \nmid m \), and we let \( M := W/U \). Now \( SL_n(q) \) is perfect and so there are no non-split modules of type \( k \cdot k \) for \( SL_n(q) \), and thus \( M \) has no trivial submodules or quotients. From Clifford Theory we find that a faithful representation of \( ASL_{n-1}(q) \) in characteristic \( p \) has dimension at least \( q^{n-1} - 1 \), since \( SL_{n-1}(q) \) has a single orbit on non-zero vectors of (the dual of) its natural module. But \( 2(q^{n-1} - 1) > \dim M = \frac{q^{n-1} - 1}{q - 1} - 2 \) and so \( M \) is absolutely irreducible. Therefore \( P \) is a uniserial module of shape \( k \cdot \tilde{M} \cdot k \).

Now \( SL_n(q) \) is perfect, and so \( \dim H^1(SL_n(q); k) = 0 \), and \( \dim H^2(SL_n(q); k) \) is the \( p \)-rank of the Schur multiplier of \( SL_n(q) \), which is 0. Now \( \dim H^1(H; k) = 1 \), which is the \( p \)-rank of \( H/H' \). The Eckmann–Shapiro Lemma then implies that \( \dim H^1(SL_n(q); P) = 1 \). A well-known long exact sequence of cohomologies, applied to the permutation module \( P \cong k \cdot \tilde{M} \cdot k \), with (trivial) submodule \( U \cong k \cdot \tilde{H} \cdot k \) and quotient \( Q \cong \tilde{M} \cdot k \) is given below. The dimensions of these cohomology groups for \( SL_n(q) \) are given below. (The dimensions for \( L_n(q) \) differ from these, and also depend on whether the \( p \)-part of \( q - 1 \) is greater than the \( p \)-part of \( n \).)

\[
\begin{array}{cccccccc}
0 & \to & H^0(k) & \to & H^0(P) & \to & H^0(Q) & \to & H^1(k) & \to & H^1(P) & \to & H^1(Q) & \to & H^2(k) & \to & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & & & a(= 1) & 0
\end{array}
\]

\(^1\)We note that the action of \( GL_{n-1}(q) \) on \( E_q^{n-1} \) is the [dual of] the tensor product of the natural representation and the determinant representation. As a result \( H \cong E_q^{n-1}.GL_{n-1}(q) \) is isomorphic to \( AGL_{n-1}(q) \) if and only if \( H \) has trivial centre, which is if and only if \( (n, q - 1) = 1 \). In the cases of interest, we have \( (n, q - 1) \neq 1 \), and thus \( H \not\cong AGL_{n-1}(q) \).
We are interested in $a := \dim \text{H}^1(\text{SL}_n(q); Q)$, and the emboldened figures, which we justified earlier, ensure that $a = 1$. There is no module $k \cdot k$ for $\text{SL}_n(q)$ and thus $\dim \text{H}^1(\text{SL}_n(q); M) = 2$. But any module $M \cdot k$ for $\text{SL}_n(q)$ actually represents the quotient group $L_n(q)$, since $k$ is not a submodule of $M \otimes k^* \cong M$. (Alternatively, note that the centraliser algebra of the module $M \cdot k$ consists just of scalar matrices. Thus central elements of $\text{SL}_n(q)$ act as scalars on $M \cdot k$, and since they act trivially on the quotient $k$ of this, they act trivially on $M \cdot k$.) Therefore we conclude that $\dim \text{H}^1(L_n(q); M) = 2$, thus completing the proof of the Main Theorem.

The corresponding construction does not work when $n = 2$ (and thus $p = 2$). This is because the module $M$ is not absolutely irreducible, splitting into two (absolutely irreducible) non-isomorphic summands of dimension $\frac{1}{2}(q - 1)$. This splitting always occurs over $\mathbb{F}_4$, and will even occur over $\mathbb{F}_2$ if $q \equiv \pm 1 \pmod{8}$. The above argument gives $\dim \text{H}^1(L_2(q); M) = 2$, and thus each constituent of $M$ has 1-dimensional 1-cohomology.

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References


