QUEEN MARY, UNIVERSITY OF LONDON

MTH6104 Algebraic Structures II

Assignment 8 **For handing in on 2nd February 2010**

Write your name and student number at the top of your assignment before handing it in. **Paperclip** *all the pages together. Hand in the assignment to my office or the undergraduate office by the due date.* **Do not place this coursework in the yellow box; I will not have access to it by that time.**

1 Prove that $\langle 2, x\rangle$ is not a principal ideal of $\mathbb{Z}[x]$.

2 Let R be an integral domain. From lectures we know the map $\psi : r \mapsto ar$ is an injective function from R to R whenever $a \neq 0$.

(a) Show that ψ is surjective if and only if a is a unit.

(b) Show that ψ is surjective whenever R is finite.

(c) Deduce that any finite integral domain is a field.

3 Let R be an integral domain. Let $T = \{\, (a, b) : a, b \in R \mid b \neq 0 \,\}$. Define an equivalence relation on T by $(a, b) \sim (c, d)$ if and only if $ad = bc$. Show that $\sim$ is indeed an equivalence relation on T. Let $F := T/{\sim}$. Attempt to define addition and multiplication on F by $[(a, b)] + [(c, d)] = [(ad + bc, bd)]$ and $[(a, b)] \times [(c, d)] = [(ac, bd)]$. Show that these operations are well-defined on F, and that F is a field under this operation. [We call F the *field of fractions* of R.]

4 Let F be a field, and let R be a subring of F containing 1_F (so that R is necessarily an integral domain). Show that if $F = \{\, ab^{-1} : a, b \in R \mid b \neq 0 \,\}$ then the field of fractions of R is isomorphic to F.

5 Let $D \neq 1$ be a squarefree integer (positive or negative). For $x = a + b\sqrt{D} \in \mathbb{Q}(\sqrt{D})$ (where $a, b \in \mathbb{Q}$), let $\bar{x} = a - b\sqrt{D}$. Define $\mathcal{O}_D := \{\, x : x \in \mathbb{Q}(\sqrt{D}) \mid x + \bar{x},\ x\bar{x} \in \mathbb{Z} \,\}$. Show that if $D \equiv 2, 3 \pmod 4$ then $\mathcal{O}_D = \{\, a + b\sqrt{D} : a, b \in \mathbb{Z} \,\}$, whereas if $D \equiv 1 \pmod 4$ then $\mathcal{O}_D = \{\, a + b\sqrt{D} : a, b \in \mathbb{Z} \text{ or } a, b \in \mathbb{Z} + \frac{1}{2} \,\}$.

6 Let $I \neq \{0\}$ be an ideal of $R := \mathcal{O}_D$. Show that I has finite index in R (in the sense that $(I, +)$ has finite index in $(R, +)$). Deduce that R is Noetherian. [The ring $\mathcal{O}_D$ is as in the previous question. For $x \neq 0$, the index of $\langle x\rangle_{\mathcal{O}_D}$ in $\mathcal{O}_D$ is $\mathrm{N}(x)$, but it may be helpful to prove a weaker result.]