Queen Mary
University of London

B. Sc. Examination by course unit 2009

MTH6104 Algebraic Structures II

Duration: 2 hours

Date and time: 6th May 2009, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorised materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): J. N. Bray and S. Majid

Question 1 Let G be a group, Ω a set, and $\omega \in \Omega$.

(a) Define what it means for G to *act* on Ω. [4]

(b) For an action of G on Ω, define the *orbit* ω^G of ω under G, and the *stabiliser* G_ω of ω in G. [4]

(c) State the Orbit–Stabiliser Theorem. [2]

(d) Now let H be a subgroup of G, and let Ω be the set of right cosets of H in G. For each $g \in G$ define a function $\pi_g : \Omega \to \Omega$ by $\pi_g : Hk \mapsto Hkg$.

(i) Prove that π is an action of G on Ω (where $g\pi = \pi_g$). [6]

(ii) Prove that the stabiliser of Hg under this action is $H^g := \{ g^{-1}hg : h \in H \}$. [6]

(iii) Determine the kernel of π. [3]

Question 2 (a) Let G be a finite group and let p be a prime. What does it mean to say that H is a *Sylow p-subgroup* of G? [3]

(b) State Sylow's Theorems (in any order). [7]

Now let $G \cong \mathrm{A}_5$, which you may assume is a simple group of order 60.

(c) Write down (without proof) the cycle types (together with their orders) of elements occurring in G, and for each cycle type state how many elements of G have that cycle type. [3]

(d) For $p \in \{5, 2\}$ determine the number of Sylow p-subgroups G, a Sylow p-subgroup P of G, and its normaliser $\mathrm{N}_G(P)$. [In the case $p = 2$ it may be helpful to identify a subgroup of $\mathrm{N}_G(P)$ strictly containing P before calculating the number of Sylow p-subgroups.] [12]

Question 3 Let G be a finite group.

(a) What is a *composition series* and *composition factor* for G? [5]

(b) State the *Jordan–Hölder Theorem* for finite groups. [4]

(c) Define what it means for G to be *soluble*. [2]

(d) Prove that if $N \trianglelefteq G$ and both N and G/N are soluble, then so is G. [9]

(e) Determine the composition factors of S_7 stating clearly any results you quote. [5]

Question 4 For each of the following statements, say whether it is true or false. If it is true, prove it. If it is false, give a counterexample, explaining *why* it is a counterexample.

(a) If $|G| = 1408 = 2^7.11$, then G has a normal Sylow 11-subgroup. [5]

(b) If G is a group with normal subgroup N such that N and G/N are abelian, then G is abelian. [5]

(c) If G is an abelian group and N is a normal subgroup of G such that $1 < N < G$, then N and G/N are cyclic. [5]

(d) If G is a finite group with subgroups H and K such that $K \leqslant H$, then $|G : K| = |G : H|.|H : K|$. [5]

(e) If G is a group and $K \leqslant \mathrm{Z}(G)$ with $K \neq G$, then G/K is cyclic. [5]

Question 5 (a) Let R and S be rings. What is meant by a *homomorphism* ϕ from R to S? [2]

(b) Prove that $\operatorname{Im}\phi$ is a subring of S. Prove also that $\ker\phi$ is an ideal of R, and that $\phi^{-1}(J) := \{\, r \in R \mid r\phi \in J \,\}$ is an ideal of R for any ideal J of $\operatorname{Im}\phi$. [10]

(c) State the First Isomorphism Theorem for rings. [3]

(d) Define what it means for a ring R to be *Noetherian*. [4]

(e) Let $\phi : R \to S$ be a ring homomorphism, and suppose that R is Noetherian. Prove that $R/\ker\phi$ is Noetherian. [6]

Question 6 (a) Let R be an integral domain, and let $r, s \in R$. Define what it means for r to be a *unit* of R; for r to be an *irreducible* of R; and for r and s to be *associates*. [6]

(b) Let R be an integral domain. Define what it means for R to be a *unique factorisation domain*. [3]

(c) Let $\mathbb{Z}[\sqrt{-10}] := \{a + b\sqrt{-10} : a, b \in \mathbb{Z}\}$, which we regard as being a subset of $\mathbb{C}$. Prove that $\mathbb{Z}[\sqrt{-10}]$ is an integral domain. (You may assume that $\mathbb{C}$ is a field.) [5]

(d) Define the norm function N on $\mathbb{Z}[\sqrt{-10}]$ by $\mathrm{N}(a + b\sqrt{-10}) = a^2 + 10b^2$ for all $a, b \in \mathbb{Z}$. (Note that $a + b\sqrt{-10} = c + d\sqrt{-10}$ implies that $a = c$ and $b = d$, so that this is well-defined.) Prove that $\mathrm{N}(kl) = \mathrm{N}(k)\mathrm{N}(l)$ for all $k, l \in \mathbb{Z}[\sqrt{-10}]$. Find all units of $\mathbb{Z}[\sqrt{-10}]$, and find all $k \in \mathbb{Z}[\sqrt{-10}]$ such that $|\mathrm{N}(k)| \leqslant 10$. [7]

(e) Prove that $\mathbb{Z}[\sqrt{-10}]$ is not a unique factorisation domain. [4]

End of Paper