Queen Mary
University of London

B. Sc. Examination 2008

MAS305 Algebraic Structures II

Duration: 2 hours

Date and time: 20th May 2008, 10:00–12:00

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

There are SIX questions on this paper.

Question 1 Let G be a group, Ω a set, and $\omega \in \Omega$.

(a) Define what it means for G to *act* on Ω.

(b) For an action of G on Ω, define ω^G, the *orbit* of ω under G, and the *stabiliser* G_ω of ω [in G].

(c) Prove that G_ω is a subgroup of G.

(d) For $g \in G$ and $\omega \in \Omega$ show that $G_{\omega.g} = \{g^{-1}hg : h \in G_\omega\}$. [The notations $\omega.g$, ωg and ω^g are equivalent.]

(e) Let the group S_3 act on itself by conjugation. Write down all orbits of this action. Pick representatives $\omega_1, \omega_2, \ldots$ for each of these orbits. For each i, determine the stabiliser of ω_i in this action.

Question 2 (a) Let x be an element of the group G. Define the concepts x^G, the *conjugacy class* of x in G, and $\mathrm{C}_G(x)$, the *centraliser* of x in G. State a theorem which connects these two concepts when G is finite.

(b) We define $\mathrm{Z}(G)$, the *centre* of G, by $\mathrm{Z}(G) := \{x \in G \mid xg = gx \text{ for all } g \in G\}$. Show the following:

(i) $\mathrm{Z}(G)$ is the disjoint union of the conjugacy classes x^G for which $|x^G| = 1$.

(ii) $\mathrm{Z}(G)$ is an abelian normal subgroup of G.

(c) Let $|G| = p^n$ where p is prime. Show that $\mathrm{Z}(G) \neq 1 [= \{1_G\}]$ unless $G = 1$.

Question 3 (a) Let G be a finite group and let p be a prime. What does it mean to say that H is a *Sylow p-subgroup* of G?

(b) State Sylow's Theorems (in any order).

(c) What does it mean for a group to be *simple*?

(d) Let $p > q$ be primes. Show that all groups of order p^3q have a normal Sylow subgroup.

(e) Let $p > q$ be primes. Show that all groups of order pq^2 have a normal Sylow subgroup.

Question 4 For each of the following statements, say whether it is true or false. If it is true, prove it. If it is false, give a counterexample, explaining *why* it is a counterexample.

(a) If G is an abelian group and N is a normal subgroup of G then N and G/N are abelian.

(b) If G is a group with normal subgroup N such that N and G/N are abelian, then G is abelian.

(c) If S is a subring of R and S has a multiplicative identity then so does R.

(d) If S is a subring of R and both R and S have multiplicative identities then $1_R = 1_S$.

(e) If R is a principal ideal domain and $a, b \in R$ then a and b have a greatest common divisor.

Question 5 Let R be a ring.

(a) What is meant by an *ideal* of R?
What does it mean for R to be *simple*?
What is meant by the ring $\mathrm{M}_n(R)$?

(b) Show that if I is an ideal of R then $\mathrm{M}_n(I)$ is an ideal of $\mathrm{M}_n(R)$.

(c) Show that if R has a 1, and J is an ideal of $\mathrm{M}_n(R)$, then J has the form $\mathrm{M}_n(I)$ for some ideal I of R.

(d) Deduce that $\mathrm{M}_2(\mathbb{Q})$ is simple.

Question 6 (a) Define what is meant by a *ring homomorphism* ϕ from R to S.

(b) Let R be a ring. What does it mean for R to be an *integral domain*?

(c) Let R be an integral domain. Define the *field of fractions* of R. [You may assume without proof that any equivalence relations you may use in your definition are indeed equivalence relations, but you must be careful to define the equivalence on the correct set.]

(d) Call this field of fractions F. Show that the binary operations on F are well-defined, and determine [without proof] the identities and inverse maps for these binary operations. Check that the additive inverse behaves as it should.

(e) What else does one have to prove in order to show that F is field?

(f) Prove that there is a subring of F isomorphic to R.

 END OF EXAMINATION