# the phrase book

basic words and symbols of higher mathematics

# 1 introduction

The language of higher mathematics has words and symbols. The most basic ones are described in this document; they are very general, and will appear in every course you take. The first year lecturers will introduce them within the context of their courses, and will expect you to use them when you communicate mathematical information, orally or in writing.

Learning the basic language of higher mathematics and being able to use it with precision and fluency is one of the main objectives of your first year at University. Your achievements in this area will be monitored by specific parts of the assessment.

# 2 logic

If $P$ and $Q$ are mathematical statements, we write

$$P \Rightarrow Q$$

to mean that *"$P$ implies $Q$"* or *"$Q$ follows from $P$"*; we write

$$P \Leftarrow Q$$

to mean that *"$P$ is implied by $Q$"*. For example

$$\begin{aligned} x = 5 \quad &\Rightarrow \quad x^2 = 25 \\ x^2 = 25 \quad &\Leftarrow \quad x = -5 \\ x^2 = 25 \quad &\Rightarrow \quad x = \pm 5. \end{aligned}$$

We use the double-headed arrow

$$P \iff Q$$

to mean that *"$P$ implies and is implied by $Q$"*. In this case $P$ and $Q$ are *equivalent statements* —they are both true or both false. The expression *"implies and is implied by"* is often replaced by the awkward *"if and only if"*: thus the statement

$$x^2 = 25 \iff x = \pm 5$$

may be read out loud as

"$x^2$ *equals* 25 *if and only if x equals plus or minus* 5",

and may also be written as

$$x^2 = 25 \text{ iff } x = \pm 5.$$

# 3 sets

A **set** is any collection of *distinct* objects. The members of a set are called **elements**, and a set is determined by its elements. In some cases, a set can be defined by listing its elements, separated by commas, between curly brackets: for example

$$\{1,2,3\}$$

denotes the set whose elements are 1, 2 and 3. Two sets are equal if they have the same elements: for example

$$\{1,2,3\} = \{3,2,1\} = \{2,1,3,1,3\}.$$

Note that the order in which the elements of a set are listed is irrelevant and repetition is ignored.

A set may be *finite* or *infinite*. The number of elements of a set is called its **cardinality**. Thus the cardinality of $\{3,4\}$ is 2.

The set $\{\}$ with no elements is called the **empty set**, often denoted by the symbol $\emptyset$. Its cardinality is zero. The empty set is distinct from "nothing" —it is more like an empty container. For the same reason, 3 is distinct from $\{3\}$, the former being a number, the latter a set having a number as its only element.

To indicate that $x$ is an element of a set $A$ we write

$$x \in A$$

which reads *"x is an element of A"* or *"x belongs to A"*. Thus

$$1 \in \{1,2,3\} \qquad 4 \notin \{1,2,3\}.$$

The symbol $\notin$ reads *"does not belong to"* or *"is not an element of"*. For example

$$7 \notin \{5,\{5,7\}\} \qquad \{7,5\} \in \{5,\{5,7\}\}.$$

(Think about it.)

A **subset** of a set $A$ is any set all of whose elements belong to $A$. We write

$$B \subset A$$

to mean that $B$ is a subset of $A$. (Some authors write $\subseteq$ in place of $\subset$.) For example

$$\{3,1\} \subset \{1,2,3\} \qquad \{2,3,1\} \subset \{1,2,3\} \qquad \emptyset \subset \{1,2,3\}.$$

A trickier example:

$$\{2,3\} \not\subset \{2,\{2,3\}\} \qquad \{\{2,3\}\} \subset \{2,\{2,3\}\}.$$

(Think about it.)

For two sets $A$ and $B$, we write $A \cap B$ for their **intersection**: this is the set comprising elements which belong to both $A$ and $B$. If $A \cap B = \emptyset$, we say that $A$ and $B$ *"are disjoint"*, or *" have empty intersection"*.

We write $A \cup B$ for their **union**: this is the set comprising elements which belong either to $A$ or to $B$ (or to both $A$ and $B$).

We write $A \setminus B$ for their (**set**) **difference**: this is the set of those elements of $A$ which do not belong to $B$.

For example

$$\begin{aligned} \{1,2,3\} \cap \{3,4,5\} &= \{3\} \\ \{1,2,3\} \cup \{3,4,5\} &= \{1,2,3,4,5\} \\ \{1,2,3\} \setminus \{3,4,5\} &= \{1,2\}. \end{aligned}$$

It is plain that the difference of two sets is unrelated to the difference of two numbers. For example

$$\{3\} \setminus \{2\} = \{3\} \qquad 3 - 2 = 1.$$

If $A$ is a set and $\mathscr{P}$ is a property possessed by some of its elements then we use the following notation to specify the set of those elements of $A$ which have property $\mathscr{P}$:

$$\{x \in A;\ x \text{ has property } \mathscr{P}\}.$$

For example, the set of all real numbers is usually denoted by $\mathbb{R}$ (see next section), so if we want to write notation for the set of those real numbers which lie between 2 and 3 then we can write

$$\{x \in \mathbb{R} :\ 2 \leq x \leq 3\}.$$

It is common to replace the colon (:) by a vertical bar:

$$\{x \in \mathbb{R} \mid 2 \leq x \leq 3\}.$$

Some authors use a semicolon (;) instead of a colon or a vertical bar.

# 4 arithmetic

The 'open face' letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are commonly used for certain infinite sets of numbers and have the following standard meanings. The set of **natural** numbers is denoted by $\mathbb{N}$. Sometimes 0 is counted as a natural number and sometimes it is not. So there are two possibilities:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\} \qquad \mathbb{N} = \{1, 2, 3, \ldots\}.$$

The set of **integers** is denoted by $\mathbb{Z}$ (from the German *Zahlen*, meaning numbers).

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

The set of **rational** numbers (ratios of integers) is denoted by $\mathbb{Q}$. The set of **real** numbers is denoted by $\mathbb{R}$. The set of **complex** numbers is denoted by $\mathbb{C}$. The two principal operations involving numbers are addition and multiplication. The **sum** of two numbers $x$ and $y$ is always written $x + y$, while their **product** may be written in several equivalent ways:

$$xy \qquad \text{or} \qquad x \cdot y \qquad \text{or} \qquad x \times y.$$

Do not confuse "$\cdot$" with the decimal point ".", e.g.,

$$3 \cdot 4 = 12 \qquad 3.4 = \frac{17}{5}.$$

The quantities

$$-x \qquad \text{and} \qquad \frac{1}{x}$$

are called the **negative** and the **reciprocal** of $x$, respectively (the latter is defined only for $x \neq 0$). The operations of subtraction and division are derived from the operations of addition and multiplication using the negative and the reciprocal, respectively. The **difference** $x - y$ and the **quotient** $x/y$ of two numbers $x$ and $y$ are *defined*, respectively, as

$$x - y = x + (-y) \qquad \text{and} \qquad \frac{x}{y} = x\frac{1}{y}.$$

To distinguish a *definition* (the right-hand side gives meaning to the left-hand side), from an *identity* (both sides are meaningful expressions, and they are equal), some authors replace the symbol "=" by one of several equivalent symbols

$$\stackrel{\text{def}}{=} \qquad \text{or} \qquad := \qquad \text{or} \qquad \stackrel{\nabla}{=}.$$

Thus the expression

$$\frac{x}{y} \stackrel{\text{def}}{=} x\,\frac{1}{y}$$

may be read out loud as

> *"the quotient of x and y is defined as the product of x and the reciprocal of y".*

Exponentiation

$$x^y$$

($x$ is the **base**, $y$ the **exponent**) is defined —in its simplest form— in terms of multiplication. Specifically, if $x$ is any number and $y$ is a positive integer, $x^y$ means $x$ multiplied by itself $y$ times:

$$x^y \stackrel{\text{def}}{=} \underbrace{x \cdots x}_{y}$$

with the stipulation that $x^0 = 1$. If $y$ is a negative integer, then $x^y$ is defined only for non-zero $x$, as

$$x^y \stackrel{\text{def}}{=} \frac{1}{x^{-y}}.$$

If the exponent is not an integer, then exponentiation can still be defined in terms of logarithmic and exponential functions, but this development lies beyond the scope of this document.

# 5 sequences

A **sequence** is a list of objects. Unlike for sets, repetition is allowed and the order is essential:

$$(1,1,2) \neq (1,2,1) \neq (1,2).$$

The elements of a sequence are denoted by a common symbol, labeled by an integer **subscript**

$$(a_1, a_2, \ldots, a_n) \qquad\qquad (a_1, a_2, a_3, \ldots).$$

In this example, the symbol is $a$ and the subscript begins from 1 (it may begin from 0, or from anywhere). The leftmost expression refers to a finite sequence, the rightmost suggests that the sequence may be infinite. A more concise notation is

$$(a_k)_{k=1}^{n} \qquad\qquad (a_k)_{k=1}^{\infty} \qquad\qquad (a_k)_{k\geq 1}.$$

The **length** of a sequence is the number of its elements. The quantities $a_1$, $a_2$, etc. (which read "*a sub 1*," etc.) are the **terms** of the sequence. If $k$ is an unspecified integer, then $a_k$ is called the **general term** of the sequence.

For example, the sequence of primes

$$P = (p_1, p_2, p_3, \dots) = (2, 3, 5, \dots)$$

is infinite, and $p_5 = 11$ (read "$p$ sub 5 equals 11"). The general term $p_k$ is the $k$th prime number.

There is some common alternative terminology. A two-element sequence may be called an **ordered pair**; a three-element sequence an **ordered triple** (note that a *pair* denotes a two-element *set*). In some cases a sequence may be called a **vector**, in which case we speak of **dimension** rather than length.

Given a sequence of numbers $(a_1, \dots, a_n)$, we may form the sum and the product of its elements

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \qquad \prod_{k=1}^{n} a_k = a_1 \times a_2 \times \cdots \times a_n.$$

The symbol $\sum$ is called the **summation** symbol. The subscript $k$ is the **index of summation**, while 1 and $n$ are, respectively, the **lower bound** and **upper bound** of summation. The index of summation is a *dummy variable*, e.g.,

$$\sum_{k=1}^{5} k^2 = \sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

The quantity $a_k$ is the **general term** of the sum. The integer sequence $(1, 2, \dots, n)$, specifying the values assumed by the index of summation, is called the **range** of summation. The symbol $\prod$ is called the **product** symbol. All terminology introduced for sums extends with obvious modifications to products.

# 6 functions

A **function** consists of two sets together with a rule which assigns an element of the second set to *each* element of the first set. The first set is called the **domain** of the function and the second set is called the **codomain**. A function whose domain is a set $A$ may also be called a function **over** $A$ or a function **defined on** $A$.

A function can be denoted by a single letter or symbol, e.g., $f$. If $f$ denotes a function and $x$ is an element of its domain then the **value of** $f$ **at** $x$, denoted by $f(x)$

is the unique element of the codomain of $f$ which is assigned to $x$ by the rule in the definition of $f$. The notation

$$f : A \to B \qquad x \mapsto f(x)$$

is used to indicate that $f$ is a function with domain $A$ and codomain $B$, which carries $x \in A$ to $f(x) \in B$. (Note the different role played by the symbols $\to$ and $\mapsto$.) For example

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto x^2$$

is the function that squares its argument, e.g.,

$$f(\sqrt{2} - \sqrt{3}) = 5 - 2\sqrt{6}.$$

The expression

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto \frac{1}{x}$$

is invalid, because $\mathbb{R}$ cannot be the domain of $f$ ($0 \in \mathbb{R}$ but $f(0)$ is undefined.) The largest subset of $\mathbb{R}$ which serves as a domain for $f$ is $\mathbb{R} \setminus \{0\}$.

Given an (infinite) sequence $(a_1, a_2, \dots)$ of elements of a set $A$, we define

$$f : \mathbb{N} \to A \qquad k \mapsto a_k.$$

The function $f$ is naturally associated to the given sequence, for $f(k)$ is the same as $a_k$. For this reason, sequences may be identified with functions defined over (subsets of) the integers.