

Calculating the Fine Structure Constant

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Calculating the Fine Structure Constant

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ABSTRACT. An introductory review of the origins of the fine structure constant problem and some of its historical connections to angular momenta in Maxwell's electromagnetic theory and to Planck's frequency to energy conversion factor h . Three formulae that give accurate values of the fine structure constant are briefly examined and one probabilistic procedure based on the Combinatorial Hierarchy that leads to an accurate value is also discussed. Then follows a refinement of analysis used by the present author in deriving a formula for the fine structure constant. The error in the value deduced earlier in relation to the experimentally obtained value was approximately two parts in one thousand million. The refinements reduce this error to approximately two parts in one *hundred* thousand million. The scheme used is called wave snapping. It involves an analysis of how a trapped wave can be bent to fit along the orbit it is following. A wave element held trapped in an orbit can be considered to be able to bend continuously with infinite snapping points so as to lie along a curved orbit or it can snap into a discrete number of linear sections so as to make a best fit to the orbit curvature. A brittleness quantum number N_b relating to the number of snaps in a constrained wave is introduced. A formula involving the integers $N = 137$ and $N_b = 25$ is derived giving a value for the fine structure constant near the center of the bounds obtained by nuclear magnetic resonance experiments.

Pour Calculer la Constante de la Structure Fine

Résumé. Une revue d'introduction est donnée pour les origines du problème de la constante de la structure fine, et quelques liens historiques entre ce problème et les quantités suivantes; moment angulaire de la théorie électromagnétique de Maxwell et la constante h de Planck (constante qui convertit la fréquence en énergie). Trois formules qui donnent des valeurs précises de la constante de la structure fine sont brièvement examinées, et une procédure probabiliste basée sur la Hierarchie Combinatoire que mène à une valeur précise est aussi traitée. Les sections qui suivent l'introduction contiennent un raffinement de l'analyse utilisée par cet auteur dans la derivation d'une formule pour la constante de la structure fine. L'erreur entre la valeur déduite plus tôt et la valeur obtenue expérimentalement était (approximativement) de deux par mille millions. Les raffinements réduisent cette erreur à (approximativement) deux par cent mille millions. La méthode utilisée s'appelle *cassure des ondes*. Elle implique une analyse de la façon par laquelle une onde bloquée peut être courbée pour s'adapter à l'orbite qu'elle suit. Un élément d'onde

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bloquée dans une orbite peut courber continuellement avec une infinité de points de *cassure* pour s'appuyer le long d'une orbite courbe ou il peut se casser en un nombre fini de sections linéaires afin d'obtenir la meilleur adaptation à la courbure de l'orbite. Un nombre quantique N_b de *fragilité* concernant le nombre de cassures dans une onde contrainte est introduit. Une formule impliquant les nombres entiers $N = 137$ et $N_b = 25$ est trouvée; elle donne une valeur pour la constante de la structure fine près du centre des limites obtenues par les expériences de la résonance magnétique nucléaire. Mots-clés et phrases. Quantique, Classique, Vacuum, Polarisation, Electromagnétique.

1.INTRODUCTION

The fine structure constant story is a key thread linking the old 19th century classical theories to the revolutionary developments of 20th century physics with all their philosophical mysteries. The fine structure constant α , strictly speaking, only emerged as a specific entity out of Arnold Sommerfeld's 1915 theory for the *fine* spectroscopic structure of the hydrogen atomic states. However, its history is intimately bound up with the concept of angular momentum though possibly not in an obvious way. The *pre-birth* connections of α with angular momentum reach back to Maxwell's time, 1864, and to the set of equations named after him. In fact, Maxwell's equations do not obviously say anything about angular momentum and so this root connection may only be apparent by hindsight. A specific amount of angular momentum arises in Maxwell's theory from the Coulomb potential in the form $\hbar_c = e^2/4\pi\epsilon_0 c$, the bar notation by hindsight from the present time, and this was to be upstaged by Planck's introduction of the quantity h of angular momentum in his energy formula $E = h\nu$ in 1900. The quantity \hbar_c only has the dimensions of angular momentum and at the time there was not in sight any actual physical angular momentum that it might represent, possibly a factor in its not being recognized as an important quantity. Planck's discovery introduced a conflict or at least a puzzling question of relationship between the existing electromagnetic theory packet of angular momentum from the Coulomb potential and his new h , they differed by the strange numerical factor $860.8 \approx 2\pi N$ with $N \approx 137$, Planck's unit being the larger. The approximate relation of this ratio to $2\pi N$ may not have been noticed at the time of Planck's discovery and even if it had been it may well have been regarded as of no significance. Planck found h by a route with practically no input from classical electromagnetic theory. He used counting arguments from the new emergent statistics, Wien's displacement law from the thermodynamics of 1893 and experimental information from black body radiation measurement studies of the time to actually determine the value that h should have. His was in the vanguard of the new physics and the fact that the new h was oddly related to the old \hbar_c would, even if noticed or even considered, have seemed irrelevant. If indeed something had to give it would have had to be Maxwell's theory because, that was being replaced by a new and more fundamental theory.

The birth of the fine structure constant occurred in Sommerfeld's 1916 paper which gave his formula for the hydrogen spectrum in terms of the quantum numbers l , n and the physical definition of α as

$$\alpha = e^2/4\pi\epsilon_0\hbar c, \quad (1.1)$$

here given in present day SI units and using $\hbar = h/2\pi$. For those aware of the Maxwell connection it would, at the stage of Sommerfeld's introduction of α , have been noticed that

$$\alpha = \hbar_c/\hbar \tag{1.2}$$

and the clear connection would have been apparent between Maxwell's theory and Planck's angular momentum, frequency to energy conversion factor h . In the years following 1916, the relation (1.2) was certainly noticed by Sir Arthur Eddington and its importance understood as can be confirmed by inspection of remarks on page 38 and indeed much of the development that appears in his famous book² *Fundamental theory*. However, it is clear that as a consequence of the indefinite nature of both of the quantities involved in that relation as actually representing *true* angular momenta it has to be admitted that this thread back to Maxwell's time was somewhat fragile. Another development in those years following the discovery of Sommerfeld's formula was to obscure it completely. This was the idea of the classical limit of the new quantum mechanics expressed in the form that quantum theory goes over into classical theory with the limit $\hbar \rightarrow 0$. This limit while giving some classical remnant from the full quantum theory clearly totally destroys the fragile connection back to Maxwell's time expressed by (1.2). The effect of this general destruction of the historical connections of the fine structure constant was to create a mystery perhaps considerable more formidable than it need have been. The so called classical limit $\hbar \rightarrow 0$ seriously damages the philosophical understanding of the quantum theory also in that it destroys the connection of quantum theory to *classical* fluid structures in which \hbar is related to vorticity. There was uncertainty about the *meaning* of α , it having lost its origins in some confusion over physical and dimensionality significance. That something of this sort had happened can be seen in the following quotation about α from the 1953 edition of Max Born's book³ *Atomic Physics*. *There seems little doubt that the existence of this dimensionless number, the only one that can be formed from e , c and h , indicates a deeper relation between electrodynamics and quantum theory than the current theories provide, and the theoretical determination of its numerical value is a challenge to physics, The solution to this problem seems to be closely connected with a future theory of elementary particles in general. All attempts have so far been in vain. The most notorious of these is Eddington, according to whom $1/\alpha$ is the value of $(1/2)n^2(n^2 - 1) + 1$ for $n = 4$, namely exactly 137. The idea that $1/\alpha$ is an integer is attractive and seems to be confirmed by the latest experiments which gave $137.030(\pm 0.016)$, very nearly integral. Yet Eddington's theory has failed to predict any new phenomenon (e.g. the different types of mesons, the Lamb-Retherford effect, etc) and is altogether too fantastic to be acceptable.* Born's remarks here surely indicate that the connection back to Maxwell had been lost while he could see the need of some such connection which would hopefully arrive in some future theory. Born severely plays down Eddington's theoretical work on fundamentals and expresses views about that work that were and still are widely held in the physics community. However, Eddington's work contains some powerful fundamental intuitive inventions that may yet prove to be of great importance in the development of our understanding of nature. The recognition of the importance of the number 137 in the quantum angular momentum context is one such original

and basic input in his work on fundamental theory. He was the originator of the idea that the whole of our physical understanding could somehow be draped around a limited number of *special* integers. His set of these special integers included 137 and as is well known the very large number $2^{127} - 1 \approx 10^{38}$, the square of this number being essentially Eddington's *number of particles in the universe*. Eddington made the identification $\alpha = 1/137$, though, unfortunately, usually referring to 137 itself as the fine structure constant. He arrived at his set of numbers by counting the components of a generalized energy tensor. Such counting can, of course, only give integers and as time passed experiment showed that α was not exactly a ratio of integers Eddington's opponents could easily discredit his theory by pointing to the fact that the fine structure constant was only *approximately* equal to $1/137$.

The counting type of argument has been greatly developed independently of the physics context to show conclusively the the set of numbers 3, 10, 137 and $2^{127} - 1 \approx 10^{38}$ are *special* and as such would likely play a key role in some systems that are *constructed* by being built up in some specific way such as by the accumulation of successive information for example. This number building construction is called the *Combinatorial Hierarchy* method. Contributors to this number theoretical system that generates these special numbers include Amson⁴, Bastin⁴, McGoveran⁵, Kilmister⁴, Noyes⁶ and Parker-Rhodes⁷. This mathematical system that is now firmly established and generates the four special numbers is a graded algebra with four levels with successively 3, 10, 137 and $2^{127} - 1 \approx 10^{38}$ elements. It should be emphasized that this construction is independent of physical theories and could supply some numerical skeleton structure supporting a variety of types of complex *system* in general. The physical world could be one such system with the numbers appearing at each level being a possible peg in the form of an *input parameter* on to which the physical theory can be hung. The physical theories could well conform to some such building principle as is implied by the combinatorial hierarchy. However, the *physical* manifestation of such integers may not necessarily appear as quantities with integral values. It would certainly not detract from the fundamental constructive role of such integers if such manifestations appeared as function of the special numbers and if this were the case the objection that Eddington's inverse α is an exact integer would surely melt away. The Born quotation does give a clear idea of the nature of the problem confronting anyone who asked the question *how can the fine structure constant value be obtained from theory*. Apparently there was no theory. There was no idea what form such a theory would take if it were found. There is no idea what form any formula that might arise would take or what it would depend on. There is also the pessimistic possibility that there was no theory hovering in the future at all and α had to be taken as an input number or a god given input condition. These are probably the reasons that so few formulae giving values for α have been found in the 78 years following *Sommerfeld's* 1916 paper. Procedures that could lead to its value but are not accompanied by a simple formula have been constructed^{8,9} and one of these which leads to a very accurate approximation to α , based on the combinatorial hierarchy will be examined in more detail.

The three solutions to this problem that have been found in the form of simple formulae are the one α_W by Wyler¹⁰ in 1969, the one α_N by McGoveran and Noyes⁵

in 1991 and one α_G by the present author¹¹ in 1994. The formulae involved are

$$\alpha_W = (9/8\pi^4)(\pi^5/2^45!)^{1/4} = 0.00729734813, \quad (1.3)$$

$$\alpha_N = (1 - 1/(30 \times 127))/137 = 0.00729735426, \quad (1.4)$$

$$\alpha_G = \cos(\pi/137)/137 = 0.00729735101. \quad (1.5)$$

Wyler's formula α_W is based on calculating the ratio of certain volumes associated with the conformal group. This group has significance in relation to relativistic wave equations and presumably this is the physical input for the Wyler formula. However, the structures of the spaces that carry this type of description are very complex and, in particular, finite volumes can only be formed with careful reference to the radii of these spaces. This aspect has been investigated by Robertson¹². There is doubt about the physical linkage of Wyler's formula and about the relevance and validity of the volumes he used. However, his formula is *simple*, only relies on a limited collection of integers and π and it is very accurate. Its error in relation to the present measured value is approximately 4 parts in 10^9 . An adverse aspect of the Wyler *type* formula is that by taking integers and π or indeed other constants to various powers to form a product with modern computers doing the calculating it is possible to find forms, often of great simplicity, that will give any prescribed number. This last remark, of course, has no bearing on the rights or wrongs of Wyler's actual result.

The formula α_N obtained by MaGoveran and Noyes is remarkable for its simplicity and its numerical accuracy. It depends essentially on integers related to the combinatorial hierarchy and its error in relation to the present day measured value is approximately one part in 10^9 . More details about α_N which was arrived at by arguing from a discrete mathematics or computer bit-string information theory like structure can be found in reference⁵.

The formula α_G obtained by the present author in 1994 is based on a *classically* based alternative¹³ to Schrödinger quantum theory which is consequently easily linked back to Maxwellian structure. The theory is a photon like *zitterbewegung* model describing states that retain locality in phase space with circular cycles of a trapped *photon* representing the usual eigenstates. The Maxwell *quanta* \hbar_c becomes a classical angular momentum quanta in phase space with quantum number 137 attached. The numerical error in α_G is approximately two parts in 10^9 approximately half the the error of Wyler's formula and twice the error of the Noyes MaGoveran formula. A large improvement on α_G will be derived in the following sections of this article.

The workers^{5,6,7,8,9} on the Combinatorial Hierarchy method have developed a scheme based on the hierarchy for calculating corrections for the physical fine structure constant first approximation $\alpha_0 = 1/137$ by appealing to an *ergodic hypothesis* and using simple probability arguments. Fully implemented the process of finding corrections becomes very involved after the first correction so that any short description could not do it justice. However, it is very promising and it appears that with ingenuity a path could be found giving the measured value of α with very great accuracy. Very briefly, the argument starts with the random selection of 137 elements and consequent probability for selecting one, $1/137$. There are 4 levels associated

with the four special integers so that an approximate value for the probability ϵ of an element not being at the first three levels under an imposed restriction can be calculated. The improved value for α is then given by $(1/137)(1-\epsilon) = 1/137.03344$. This is a great improvement obtained relatively easily. These steps are then refined by taking account of subtle linear independence questions at level 2 and the procedure is very involved from thereon. The next correction gives the value $1/137.03652$ with much improved error rating and subsequent corrections reduce the error still further to greatly lower than the error of Wyler's formula. This is a very interesting scheme constructed around the combinatorial hierarchy, involving great minimality of physical input with mostly probabilistic like considerations. The details of this scheme can be found in references^{8,9}.

2. WAVE CAPTURE

The rest of this paper is concerned with a substantial extension of the work¹¹ where the present author derived a formula for the fine structure constant from his alternative theory¹³ for the quantum process. There it was shown that the assumption that classical theory contains the more refined unit, \hbar_c discussed in the introduction, of angular momentum than the \hbar or $\hbar/2$ of the orthodox theory leads to the formula

$$\alpha_G = \cos(\pi/N)/N \quad (2.1)$$

for the fine structure constant in terms of an *integer* parameter $N = 137$. It was shown that, when translated into rotating length magnitudes, the *refined* angular momentum assumption implies the simple formula for α in terms of the angle $\chi_N = \pi/N$ and the integer $N = 137$ acting as a *classical* quantization number. This result can be read off from the wave capture diagram figure 1. The formula (2.1) gives the value of the fine structure constant α to very great accuracy when N is set equal to 137. $\alpha = \cos(\pi/137)/137 \approx 7.297351 \cdot 10^{-3}$. The error in this value in relation to the latest experimental value¹⁴ for α is of the order of two parts in one thousand million.

The author's alternative theory does not generate the actual numerical value $N = 137$ of the integer N . However, the theory applied to the general eigen-state *geometrical* structure in a phase space does imply that an integer N should exist so that a radial quantization $r = Nl_c$ occurs with radial length quanta of size l_c equal to one classical electron radius. This then implies a circumferential quantization in the same phase space with angular quanta of size π/N . Once the formula (2.1) in terms of N is found, it becomes clear that N should have the value 137 if the formula is to agree with experiment. The author has chosen the nuclear resonance experiment range as a target for a *theoretical* derivation of the value of α because although many different experiments were taken into account and very elaborate statistical analysis was applied in arriving at the recommended values¹⁴ for the physical constants, the very close limits for the value of α , (± 33) at the 7th and 8th decimal place in units of 10^{-3} , are attributed to the extreme accuracy of the magnetic nuclear resonance experiments under QED interpretation.

Experimental information¹⁴ concerning the value of α and the value from the author's formula $\alpha = \alpha_G$ is exhibited in table 1. The value of α_{ex} is calculated

from the defining formula $\alpha = e^2/4\pi\epsilon_0\hbar c$ using the experimental values for e and \hbar and with the values of ϵ_0 and c assumed to be exact.

Table 1 Various *nearby* measures associated with the value of α

Magnitude order	Name	Value	Source
1	α_{exmin}	0.00729734425	Raw experiment ¹⁴
2	α_G	0.00729735101	Author's theory ¹¹
3	α_{Nucmin}	0.00729735275	NR with QED ¹⁴
4	α_{ex}	0.007297353077	Raw experiment ¹⁴
5	α_{Nuc}	0.00729735308	NR with QED ¹⁴
6	α_{Nucmax}	0.00729735341	NR with QED ¹⁴
7	α_{exmax}	0.00729736190	Raw experiment ¹⁴

In this paper, the argument giving the result (2.1) is refined so as to give a modification of this formula which improves its accuracy in relation to the measured value for the fine structure constant, α_{ex} or α_{Nuc} , by a factor of about one hundred. This level of accuracy takes the theoretical value near to the center of the limits that are claimed by the experimental nuclear magnetic resonance technique workers. The procedure that is employed in this research is most easily explained in terms of the *wave capture diagram* figure 1 which represents important details of the substructure in phase space of a quantum state which has an amount of energy E numerically the same as the rest energy $E = m_0c^2$ of an electron. Very briefly the salient features of this diagram and what it represents will now be described. The diagram represents the alternative theory equivalent of an eigenfunction in orthodox Schrödinger theory. The diagram is built as a *quantization* construction from an integer number, N , of classical electron radii linearly added to make the smaller circle radius which is also regarded as the inner radius of the *zitterbewegung*^{15,16,17} state. This terminology arising from random aspects of the maximal light speed c movement of a wave attached to the perimeter of the larger circle which is also regarded as the outer radius of the state. The larger radius has the magnitude $L_0 = 2l_0$, this being the same as the magnitude of one electronic Compton wave length divided by 2π . The main physical principle involved in this kinematic geometric construction is that of the relativistic length contraction effect which makes possible the trapping of a wave of rest length $L_0 = 2l_0$ within a the much smaller element of length $L = (1 - (v/c)^2)^{1/2}L_0$ between the points B_2 and A_1 . The speed $v = \cos(\pi/137)c$ in this formula is very large due to the smallness of the angle $\pi/137$. A more detailed explanation of the significance of this diagram can be found in references Gilson^{11,13}.

because its shape is very relevant to the position freedom of the trapped wave and the limit procedure to be introduced to locate the best position.

Thus the T -region plays the key role in the following analysis of how the detailed implementation of the placing of the trapped wave within its confines determines the value of the fine structure constant. In figure 2, six curve segments between the end segments A_1A_2 and B_1B_2 of the T -region have been indicated with the plain numbers 1,2,3,4,5,6 and in that same order with regard to increasing curve length. We shall denote the lengths of these six finite curve segments by $s_1, s_2, s_3, s_4, s_5, s_6$ respectively. The currently most accurate experimentally determined value for the fine structure constant using nuclear magnetic resonance and QED corrections is recorded¹⁴ as $\alpha_{Nuc} = 0.00729735308$. This value would occupy a position just below s_2 on the table and some substantial way above s_3 on the table. Thus the value of α_2 derived from the authors formula via s_2 is the best value from the point of view of physical agreement displayed in the table. The higher values displayed lower in the table are not so good but they are all bounded by the best known approximation to α which is $\alpha_6 = 1/137$ at the bottom of the table. This particular value played a central role in Eddington's *Fundamental Theory*^{2,18}. All the values agree up to the 5th decimal place. That is their error rating overall is approximately less than 2 parts in 10^6 so that under *usual* accuracy standards they are all good approximations. However, the physical measurements demand that we work to the extraordinary accuracy standard of at least 2 parts in 10^8 under raw physical measurements or 6 parts in 10^{10} if the nuclear QED interpreted measurements are taken seriously. Clearly, these are very demanding constraints on the analysis that will be needed to improve on the formula proposed by the author. This is given by α_2 and differs from α_{Nuc} by approximately 2 parts in 10^9 , thus conforming to the raw measurement constraint but remaining outside the nuclear resonance constraint. However, a very substantial improvement will be delivered in the following sections of this paper which will rectify this deficiency.

to add this physical dimension to the mathematical curve α -family structure. Here the idea of brittle wave bending replaces that of smooth bending flexibility. The brittle wave idea will be described in detail later but first it will be explained why a family of waves of a specific form can be used to improve on the original formula (2.1).

The derivation of that formula rests heavily on the relativistic contraction effect with speed $v = c \cos(\pi/137)$. Referring to the diagram figure 1, it is recalled that the original argument used employs the relativistic¹⁹ contraction factor $(1 - (v/c)^2)^{(1/2)}$ to make the assertion that this factor reduces the length $2L_0 = 4l_0$ to the smaller length $2L$ between B_2 and B_3 in virtue of the tangential velocity v of the point A_1 at the segment center. This is very accurately true because of the smallness of the moving element B_2B_3 but it is not exact because of the rotational motion of the element. Thus there is a sense that the straight line B_2A_1 is possibly not the best curve segment to represent the trapped wave. It will be seen in the next section that the arc B_1A_1 is a better candidate and one reason for this is that the speed of motion all along this *curve* segment is exactly v so making the relativistic contraction factor rigorously applicable. However, we shall also see that this *smooth* curved segment has a disadvantage. These remarks thus motivate the search for a curve with optimum properties as the candidate for the trapped wave status. It will be shown that an optimum curve segment can be found in the family of curves to be introduced.

The idea of waves placed around the circumference of a circle was quite extensively and effectively used in the early days of quantum theory by De Broglie, Born and others to give a geometrical interpretation to the quantum conditions of Bohr's atomic theory. The idea was essentially that a single definite wave form is only possible round the circumference of a circle when the circumference is a whole number n times the wave length λ . Thus the key equation involved in applying this idea was $n\lambda = 2\pi r$ and the values of n obtained in this way correlated with the quantum number n from Bohr's theory uniquely. That approach to the analysis of the quantum orbital geometry was the beginning of the recognition of the importance of continuity which was there being represented by the idea that the n waves placed round the circle would join up continuously to make for single valuedness and uniqueness. However, the idea that a one dimensional wave of wave length λ on a straight line element can be distorted so as to be placed exactly along a curve of *length* λ involves the implicit assumption that physical waves can be continuously and smoothly bent so as to fit a curve exactly. As a result of the success of the deformation idea and its considerable degree of success and accuracy in the early days quantum context questions of the physical validity of the assumption do not seem to have been considered or possibly even noticed. The idea to be applied here involves the somewhat converse assumption that in the quantum context nature does not know about smoothly deformable waves.

Thus we introduce idea of the *brittle* wave that snaps into line segments in order to conform to a curved contour. On the basis of the brittle assumption, the curve segments that should be included in the α -family as being possible candidates to represent the physical wave regarded as trapped in the T -region should be continuous but composed of *linear* segments only, each linear segment joined to its neighbors by end points where a *snap* has occurred in the unbent full linear wave

length segment. The snaps are necessary if the full wave length is to conform in some approximation to a smooth curve. The idea is not that the wave segment that snaps at some point is broken. Rather a single snapped linear segment will be considered to become two linear segments joined at the snap point and rebonded at a definite snapping angle. We shall consider in more detail the amount of deviation that can occur at such a snap. Firstly referring to our six candidate curves in the diagram figure 2, we see that s_6 and s_2 our *best* candidate so far should be excluded from the α -family because they are not straight segments but rather smooth as well as continuous and would have had to be continuously deformed from a pure line segment. We can also disregard s_4 , s_5 and s_6 because although they might be regarded as having historical interest they lie well outside the recent¹⁴ raw experiment bounds which are $\alpha_{exmin} = 0.007297344254$ and $\alpha_{exmax} = 0.7297361900$ as given in table 1. The segments s_1 and s_3 are also outside this range but we shall retain them and use them as the lower and upper length bounds to the α -family. We retain them because they supply a strong clue concerning the amount of snapping angle that is physically allowable. Inspection of the s_1 and s_3 segments lengths reveal the simple relation $s_1 = s_3 \cos(\chi/2)$, showing, as it should, that s_1 is slightly less than s_3 , $\cos(\chi/2)$ having the value 0.99993427 slightly less than unity. The two extreme bounds have the property of sandwiching the value s_2 together with the physically important segment we seek. The $\cos(\chi/2)$ in the relation between these bounding segment lengths is the cosine of the angle $\chi/2$ that projects the s_3 length onto the s_1 length. The experimental evidence is that the target curve segment is some small amount in length *definitely* above the s_2 value. It should be noted that s_1 and s_3 are the chord and a central tangent to the circle defined by the $\chi = \pi/137$ sector. It will be shown that a sequence of new ever closer bounds within the extremes can be defined having this same property. Each pair of closer bounds in the sequence will be correlated with an integer which will be denoted by N_b . The subscript b refers to brittleness and N_b denotes the position in the sequence of closing bounds and, in a sense to be explained, the minimum brittleness of curves within the bounds being considered.

Consider the division of the $2\pi/137$ circular sector between B_2 and B_1 into N_b smaller circular sectors of angular size $2\pi/(N_b 137)$, the number N_b taken to be an integer. Each one of these smaller sectors will have a chord and central tangent with lengths given by

$$s_{1,N_b}/N_b = Nl_c \sin(\chi/N_b) \quad (3.1)$$

and

$$s_{3,N_b}/N_b = Nl_c \tan(\chi/N_b) \quad (3.2)$$

respectively. Clearly the sum of these chord lengths and the sum of the tangent lengths for the $\chi = \pi/137$ half sector give the lower s_{1,N_b} and upper bounds s_{3,N_b} at sequence number N_b . The sequential bounds satisfy the relation $s_{1,N_b} = s_{3,N_b} \cos(\chi/N_b)$. Thus the term $N_b = 2$ gives the relation noted earlier between the extreme bounds. We have, in effect, constructed polygonal inner and outer approximation to the circular perimeter segment s_2 and by taking N_b sufficiently large we can get arbitrarily close to that segment. The number of sides of such a polygon enclosing the *whole* orbit will be NN_b and so the exterior angle of the polygon will be $2\pi/NN_b$. This angle can be taken to be the wave *snapping* angle

$\chi_{s,N_b} = 2\pi/NN_b$. The snapped wave which occupies only the π/N angular sector will be composed of $N_b/2$ of the polygon's sides. This can involve half a side if N_b is odd. The physical significance of the sequential pairs of bound is that with increasing N_b , the set of curves contained becomes increasing relativistically valid. This is because the closer they are snapped to the smooth curved segment s_2 by increasing N_b the more accurately do they move along their own length with the velocity v . However, we do not wish to get arbitrarily close to the circular segment but rather we wish to remain in length just above its length so that

$$s_{3,N_b}/(\pi 2l_0) = \alpha_{ex}, \quad (3.3)$$

the best experimental value. Thus a constraint on smoothness competes with relativistic validity to generate an optimum result. Thus, as we have a formula for the function s_{3,N_b} of N_b , equation (3.3) can be used to find the value of N_b we seek. The number $(N_b - 1)/2$ will then be, according to our supposition, the number of internal snaps that nature can tolerate in bending a wave so as to conform to a circular contour with the best relativistic configuration allowable under this constraint. If we seek solutions of equation (2.3) for values of α_{ex} in the nuclear resonance QED range $(\alpha_{Nucmin}, \alpha_{Nucmax})$, we find the four integer solutions for N_b given in table (3)

Table 3 N_b solutions with α values and NR bounds

N_b	α -values
	$\alpha_{Nucmax} = 0.007297353410$
24	$\alpha_{3,24} = 0.007297353232$
	$\alpha_{Nuc} = 0.007297353080$
25	$\alpha_{3,25} = 0.007297353057$
26	$\alpha_{3,26} = 0.007297352903$
27	$\alpha_{3,27} = 0.007297352766$
	$\alpha_{Nucmin} = 0.007297352750$

From the formulas (3.2) and (3.3) the general dependence of the α -family on the number N_b is given by the formula

$$\alpha_{3,N_b} = s_{3,N_b}/(2\pi l_0) = \alpha_2 N N_b \tan(\chi/N_b)/\pi \quad (3.4)$$

with

$$\alpha_{3,25} = s_{3,25}/(2\pi l_0) = \alpha_2 25 N \tan(\chi/25)/\pi. \quad (3.5)$$

The values given by the four N_b values are displayed in the table. The value of $\alpha_{3,25}$ differs from α_{Nuc} by $2.3 \cdot 10^{-11}$ and is by far the best option. The number N_b was defined over the full $2\pi/137$ sector. The trapped waves only occupy the half sector so that the number of snaps N_s in a trapped wave will be $N_s = (N_b - 1)/2$. Thus for the best option given in the table $N_b = 25$, we conclude that nature can tolerate 12 snaps in a trapped wave as acceptable in its bending to fit a circular orbit segment. It is convenient to denote the correction factor in (3.5) by β . Thus,

$$\beta = 25 N \tan(\chi/25)/\pi \quad (3.6)$$

and then our final formula for the brittle wave generated formula for the fine structure constant α_b can be put into the tidier form

$$\alpha_b = \alpha_2 \beta. \quad (3.7)$$

The correction factor as it appears in (3.4) depends on N_b and in the limit of $N_b \rightarrow \infty$ the factor goes to unity with $\alpha_{3,\infty} = \alpha_2$. This is the smooth curve limit that is accurate relativistically but takes no account of the brittleness limitation. In that the structure of diagram 1 generates α_2 in the first instance, it is strictly relevant to that same limit and can only be interpreted as approximate in the more general brittle wave context. We can confirm this and discover how its interpretation should be modified by considering the 137 quantization equation that pertains in the more general case of brittle trapping. As α_2 can no longer be equal to $l_c/2l_0$ as was implied in the original diagram 1 we should restore the unavoidable relation $\alpha_b = l_c/2l_0$ for the new improved fine structure constant at the $N_b = 25$ physical level. The new quantization condition becomes

$$137(l_c/\beta) = 2l_0 \cos(\pi/137). \quad (3.8)$$

This shows that in the general case the length quanta should be taken to be $l_b = l_c/\beta_{NN_b}$ or $l_b = l_c/\beta$ in the physical limit, this last quantity being called the brittle

classical electron radius. For the general case this also means that wherever l_c occurs in figure 1 or table 2 it should be replaced with l_b . The original steps in deriving values for α still hold because $l_b/(2\pi l_0) = l_c/(\beta\pi 2l_0) = \alpha_b/(\beta\pi) = \alpha_2/\pi$ as before. The velocity $v = \cos(\chi)c$ of the inner circle is not changed because in the brittle wave case $v_b = Nl_b\omega = Nl_b c/2l_0 = Nl_c c/\beta 2l_0 = N\alpha_b c/\beta = N\alpha_2 c = \cos(\chi)c = v$ as before. We note the approximation to the correction factor as a function of NN_b

$$\beta_{NN_b} \approx 1 + (\pi/NN_b)^2/3. \quad (3.9)$$

At the value $NN_b = 137 \cdot 25$ this is indistinguishable up to 12 decimal places from β which has the value $\beta = 1.0000002805$. The closeness to unity of this factor indicates vividly the highly refined and subtle nature of the correction to the formula α_2 that has been obtained. Finally we note that the approximation for α_b can be expressed in terms of the first three special numbers $k_1 = 3$, $k_2 = 10$, $k_3 = 137$ of the combinatorial hierarchy using $25 = k_2^2/4$.

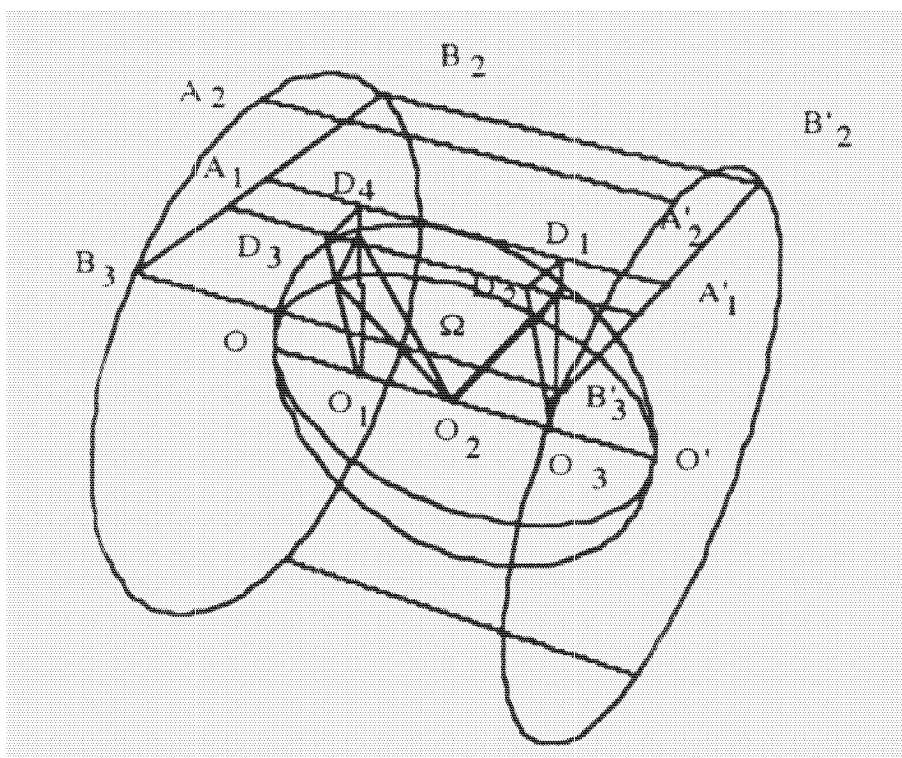
$$\begin{aligned} \alpha_b &= e^2/4\pi\epsilon_0\hbar c \\ &= (k_2^2/4\pi) \cos(\pi/k_3) \tan(4\pi/k_3 k_2^2) \\ &\approx \cos(\pi/k_3)(1/k_3 + (4\pi)^2/k_1 k_3^3 k_2^4). \end{aligned} \quad (3.10)$$

The appearance of the 4π in the formula for α_b seems to suggest that if the original argument in the derivation of the formula for *alpha* had been three dimensional the number $k_2^2 = 100$ might have arisen naturally as the measure of some number of solid angles of specific size. This can be confirmed by a two stage procedure partially illustrated by figure 3. The first stage relates the two dimensional wave trapping of a linear wave between two circles to a similar three dimensional trapping of a rectangular wave between two cylinders. the second stage involves relating the trapping angle $\chi = \pi/137$ to a corresponding trapping *solid* angle Ω_t associated with the rectangular wave. These are related by $\Omega_t = \chi = \pi/137$. Consequently it is possible to define a solid angle quanta Ω_q with the property $k_2^2 k_3 \Omega_q = 4\pi$. The solid angle Ω_q divides a spherical surface into $k_2^2 k_3$ *fundamental* areas and the formula (3.10) for α_b acquires a clear three dimensional interpretation. Figure 3 illustrates steps in this projection procedure associated with the *D*-rectangle. In the diagram, an area on the central spherical surface at center O_2 and radius Nl_c bounded by two great circles and two end curves is cylindrically projected from the line $O_1 O_3$ on to a region of the same area on a central cylinder not shown also of radius Nl_c and thence onto the area denoted by the letters D lying on the plate $B_2 B_3 B'_3 B'_2$. The spherical area with the solid angle Ω centered at O_2 is thus related to the *D*-rectangle on the plate. If $D_4 D_1 = A_1 A'_1/2$, the solid angle $\Omega = \angle D_1 O_3 D_2$.

4. HYDROGEN-137 AND OTHER ASPECTS

It is appropriate to address some important and interesting issues raised by the referees when reviewing the first draft of this article. The first issue is what further rationale can be made for the appearance and use of the integers 137 and the additional integer 25 that has been introduced in deriving the improved formula for α . With regard to 137 we shall here make a substantial step in identifying

FIGURE 3 RECTANGULAR WAVE CAPTURE IN CYLINDRICAL SECTOR



its numerical value $N = 137$ with the number of positive charges on a specific theoretical hydrogen like atom. The formula for the fine structure constant (2.1) was obtained by the present author and published in an article¹¹ in 1994 using his alternative theory¹³ for the quantum process and the key diagrammatic representation for states in that theory called the *the wave trapping diagram*, figure 1 in this article. All the states in the alternative theory conform to this diagram which describes a *zitterbewegung*^{15,16,17} motion at the speed of light. The probabilistic aspects of this theory arise from assemblies of such circulating state representations. The circulation motion takes place in a *phase* space involving movement in a dimension orthogonal to the ordinary three dimensional configuration space in which the orthodox Schrödinger theory activity is considered to take place. Thus it seemed that the formula could only have a geometrical representation in this hyperspace and could thus only be derived in that context. This would account for its not having been obtained from the orthodox theory in the 78 years following Sommerfeld's discovery¹ of expression (4.1) involving the fine structure constant which was thereafter denoted by α . However, now that this formula for α is known, the position has changed and in searching in the orthodox theory we know what

to look for and also have some idea where we might look if a derivation is to be obtained within the usual version of Schrödinger theory. Fortunately, it is possible to identify one *theoretical* state that can occur in the Schrödinger theory repertoire of hydrogen like atoms that is conformable in structure and meaning to the wave capture diagram. The agreement between this state and the diagram is sufficiently close for the state to be mapped onto the diagram so that the derivation required can be generated. This rare state is associated with an electron moving in the first Bohr orbit of a nucleus with $Z = 137$ positive charges. Clearly this is well outside the physical range but is a *theoretically* well understood state. The velocity with which the first Bohr orbit in this state is described is well known theoretically. It coincides numerically with the general trapping velocity for all states in the alternative theory. This fact and the related fact that the electronic motion takes place on a circular orbit makes possible the interpretation of the structure of this state in terms of the wave capture diagram so making a derivation using orthodox theory possible. Neither the original derivation nor the present one gives a reason for the appearance in the formula of the particular integer value $N = 137$. However that N should be an integer was a basic feature of the original derivation in that it represented a *classical* quantization number of *length* quanta. It will be clear in the following that the actual integer $N = 137$ that arises in the present derivation is in fact the number of positive charges on a specific *theoretical* atomic nucleus. However, this still does not *derive* the actual integer value 137 but it does give it a specific identification within the orthodox quantum theory and indeed conclusively confirms the validity of the formula (2.1).

Consider the Sommerfeld formula for the energy of hydrogen like atoms with a Z -valued positive charge Coulomb central field,

$$E_{n,j,Z} = m_0c^2 \left(1 + \left(\frac{Z\alpha}{n - j - (1/2) + ((j + (1/2))^2 - (Z\alpha)^2)^{1/2}} \right)^2 \right)^{-1/2} \quad (4.1)$$

with n what is usually called the principal quantum number and j with its possible half integral values is the quantum number for total angular momentum. The expression (4.1) is not the last word in energy specification for hydrogen like atoms because it does not include the Lamb shift complications. However, it is extraordinarily accurate and serves here to introduce the fine structure constant α in relation to the nuclear charge number Z . The energies of the first Bohr orbits for various values of Z are given by $n = 1, j = 1/2$ and $1 \leq Z \leq 137$ is

$$E_{1,1/2,Z} = m_0c^2(1 - (Z\alpha)^2)^{1/2}. \quad (4.2)$$

In the physical context, values of Z up to 103 describe physically well known hydrogen like atoms. The values above $Z = 103$ describe theoretically possible hydrogen like atoms which generally speaking have not been physically detected. We shall make use of the *theoretically* allowable extreme hydrogen like atom for which $Z = 137$. We can study this special case without actually referring to its energy function but rather by making use of just two fundamental results from the orthodox treatment of the hydrogen family and the centrifugal force equilibrium

equation together with just a little special relativity theory. The results required are derived from Schrödinger equation theory under the Coulomb potential

$$V = -Ze^2/4\pi\epsilon_0 r. \quad (4.3)$$

and are the values of the velocities $v_{B,Z}$ for $Z = 1, 137$ in the first Bohr orbits where $n = 1$. They are found by standard Schrödinger theory to be

$$v_{B,1} = \alpha c \quad (4.4)$$

and

$$v_{B,137} = 137\alpha c. \quad (4.5)$$

The centrifugal equilibrium equation expresses a relation between the Bohr orbit radius $r_{B,Z}$ and the Bohr orbit velocity $v_{B,Z}$ and is

$$Ze^2/4\pi\epsilon_0 r_{B,Z} = m_0 v_{B,Z}^2. \quad (4.6)$$

Using the classical electron radius l_c given by

$$e^2/4\pi\epsilon_0 l_c = m_0 c^2 \quad (4.7)$$

we get the relation,

$$Zl_c/r_{B,Z} = (v_{B,Z}/c)^2. \quad (4.8)$$

Let us define

$$\alpha_{B,Z} = v_{B,Z}/cZ \quad (4.9)$$

and use (4.8) to get the relation

$$r_{B,Z} = l_c/Z\alpha_{B,Z}^2. \quad (4.10)$$

We see from (4.5) that because 137α is very close to unity $v_{B,137}$ has the same, very close to c , value that the trapping velocity v has in the author's alternative theory¹¹ which is used in the wave trapping context. This is the reason that this special extreme state from orthodox Schrödinger theory can be related to the wave trapping diagram from the alternative theory. However, the wave trapping diagram is used in the context of the alternative theory in a *phase space extension* of ordinary configuration space, while we are about to use it here in the context of the orthodox theory totally in ordinary configuration space. The wave capture diagram figure 1 is our key analytic tool. It should be noted that because of the smallness of the angle χ_N and the smallness of the radii difference $r_0 - r$, the figure has to be greatly distorted in order for it to be readable. Referring to fig 1, consider the circular first Bohr orbit for a $Z = 137$ hydrogen like atom. Suppose a wave in this orbit is trapped in the linear segment of size L by the relativistic¹⁹ contraction effect so that

$$L = r_{B,137}(1 - (v_{B,137}/c)^2)^{1/2} \quad (4.11)$$

with $r_{B,137}$ the radius of the orbit being taken as the *rest* length of the wave. A complete wave *statically* trapped and lying along the *radius* $r_{B,137}$ in such an orbit

would be equivalent to a whole circumferential wave length $2\pi r_{B,137}$. With $n = 1$ this would mean a single Compton wave length, in this case $\lambda_C = h/m_0c$ with the uncrossed variety of h , wrapped once round the circumference but not in motion. The wave of length L is greatly contracted but is in motion with velocity $v_{B,137}$ and is a relativistic dynamical representation of the same wave. From (4.4),(4.5) and (4.9) it follows that $\alpha = \alpha_{B,1} = \alpha_{B,137}$. Thus using (4.10) we can write (4.11) as

$$L = (l_c/(137(\alpha)^2)(1 - (137\alpha)^2)^{1/2}. \quad (4.12)$$

In the wave capture diagram applied in this orthodox context we identify $r = l_c/\alpha = 2l_0 = \text{radius of inner circle}$ and $r_0 = r_{B,137} = \text{radius of outer circle}$ with

$$\tan(\chi) = L/r = \frac{(1 - (137\alpha)^2)^{1/2}}{137\alpha}. \quad (4.13)$$

It follows from (4.13) that

$$\alpha = \cos(\chi)/137. \quad (4.14)$$

The circumference of the inner circle is

$$2\pi l_c \alpha^{-1} = 2\pi 137 l_c / \cos(\chi). \quad (4.15)$$

Thus stripping away the integers and regarding them as *quantization* numbers gives the circumferential length *quanta*

$$C_Q = \pi l_c / \cos(\chi) \quad (4.16)$$

and this will be contained in the angular segment χ if

$$r\chi = l_c \alpha^{-1} \chi = \pi l_c / \cos(\chi). \quad (4.17)$$

We note that the radius of the inner circle is

$$2l_0 = l_c \alpha^{-1} = 137(l_c / \cos(\chi)). \quad (4.18)$$

Thus, in this context, the radial length *quanta* are

$$R_Q = l_c / \cos(\chi). \quad (4.19)$$

We also note from (4.10) that

$$r_{B,137} \cos(\chi) = 2l_0 = 137R_Q. \quad (4.20)$$

This is a *special* direction length projection quantization formula applicable to this quantum state and is analogous to similar formulae usually involving angular momentum. Using (4.14) it follows from (4.17) that

$$\chi = \frac{\pi \cos(\chi)}{\cos(\chi)137} = \pi/137. \quad (4.21)$$

Finally from (4.14) and (4.21)

$$\alpha = \cos(\pi/137)/137. \quad (4.22)$$

The formula for the fine structure constant (4.22) has here been derived from the usual Schrödinger theory for states controlled by the Coulomb potential (4.3) by a very simple argument. The validity of the formula is thus strongly confirmed and we now also have obtained a definite anchor for the number 137 as representing that same number of electronic charge units. The possibility of the *two* kinematic types of description for the same state that are involved in the derivation, the static *spread out* one and the relativistic *contracted* one, describing the rotating electron fits well with the *duality* idea. In the former type the rotating electron is entirely wave like and in the latter type it is more particle like.

The author²² has also shown that it is possible to derive rigorously, easily and by elementary methods Sommerfeld's formula (4.1) by using the base formula (4.22) within a simple generalization of the direction projection quantization idea that is used in the above derivation of the base formula (4.22). This is thus yet a further validation of the correctness of the base formula (4.22) and which also shows that it has the same error level as Sommerfeld's formula.

Both McGoveran and Kilmister calculate 137 as $2^2 - 1 + 2^3 - 1 + 2^7 - 1$ each using different lines of argument resting on the combinatorial hierarchy. The detailed identification of this number as referring to electromagnetism is different in the two cases. This difference results in different calculations being used to get improved versions of their formulae. Monumental contributions towards finding an accurate value for α have been made by Kinoshita and Lindquist²⁰ via the orthodox QED formalism working with eighth order magnetic moment QED contributions. Their most recent value for α^{-1} is 137.03599222 or $\alpha = 0.007297352935$ which is obtained from information about the anomalous magnetic moment of the electron. It is suggested that this value is to become the recommended value in the next adjustment of the fundamental physical constants. A detailed discussion of many aspects of the relations between the fine structure constant and other fundamental constants and the current research into refinements can be found in a comprehensive contribution by B.N.Taylor²¹ to the subvolume b of *Fundamental Constants in Physics and Chemistry*.

The author does accept that the integer 5^2 which occurs in the refinement formula (3.5) is in the first instance a *fitting* parameter with a value given so as to make the best fit. However, a well defined geometrical parameter which is part of a clear theoretical construction is being fitted to a recommended value which is itself ascertained from a complex combination of both experiment and theory. This is in accord with *usual* practice in theoretical physics. One of the referees has pointed out the interesting fact that 5 occurs in the wave length form of the Wien displacement formula. In view of the closeness in conceptual area of that formula to the work in hand this does seem to be a significant point.

5 CONCLUSIONS

The improvement in the theoretical formula and value for the fine structure constant derived in this paper largely arises from a refinement in the accuracy of relativistic conformity of the construction used. The important contribution of Lorentz contraction in the argument is improved by introducing the brittle wave concept. The limit process homes in on those waves which are relativistically ideal smoothly curved, infinitely brittle and which have a unique single velocity at all their parts. However, in the limit process the *physical* waves of importance are arrived at before the ideal is encountered because of the what appears to be nature's quantum domain brittleness limit at $N_b = 25$. In terms of a whole circular orbit the brittleness limit can be seen as a characteristic property at the atomic level that nature's best circle is a polygon with $n = 137 \cdot 5^2$ sides.

The merit of a formula like (2.1) as giving the value of a physical constant lies in the four aspects of plausibility of physical derivation, extreme numerical accuracy, great simplicity and dependence only on *integers* and other precise mathematical constants, in this case π . The enhanced formula (3.7) is more complex but not greatly. Given the great accuracy of α_b it is likely that the best solution to the problem of finding a formula for the fine structure constant has been obtained. Given the unlikely event that future measurements change the physically recommended value for α substantially, the present theory can adapt by focusing on a different value for the brittleness number N_b . Very *small* such changes are more likely and in particular, if the recommended value is taken to be the Kinoshita Lindquist value mentioned in section 4, then by taking $N_b = 26$ we can get $\alpha_{3,26}^{-1} = 137.03599280$, or from table 3, $\alpha_{3,26} = .007297352903$ which differs from the Kinoshita value by approximately $3.2 \cdot 10^{-11}$. Although $N_b = 26$ does give a very accurate agreement with Kinoshita's value we lose out on the connection with the combinatorial hierarchy numbers exhibited in equation (3.10).

However, the problem that still remains is why $N = 137$ and $N_b = 25$. Both of these integers have the character of being sub-quantum eigenvalue like quantities. Both arise as feature of the author's *classical* alternative theory. Strictly N measures the number of *brittle* classical electronic radii that make up the inner radius of the state with energy $E = m_0 c^2$ and $N_b/2$ measures the number of sub-angular sectors that make up the $\pi/137$ angular slot that holds the zitterbewegung state in its circular motion. However, although the meaning of these integers is clear the theory does not supply a reason for their particular values. With regard to the integer 137 there is Eddington's work² in which the dimensions of a phase space helps give a definite significance to its value through the number of components of a generalized energy tensor. As remarked earlier, great progress in generating special numbers including 137 has been achieved by Bastin⁹ and Kilmister¹⁸ using the *combinatorial hierarchy method*.

Apart from the suggestion for a significance of the integer 5 supplied by one of the referees, as far as the author knows, a special significance for the number 25 does not seem to have emerged from any alternative direction. However, this seems to be offset by the connection to the three dimensional rectangular wave trapping case when $k_2^2 = 100 = 4N_b$ becomes a controlling integer in the formula for α_b . The value of the fine structure constant plays an important role in the question of the

variability of fundamental constants with cosmological epoch. The form of α_b only depending on fixed integers and π would seem to rule out any continuous variation with time. Jumps, of course, could not be excluded. These aspects are discussed in some detail in Narlikar's book²³.

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