Implicit Differentiation

Consider a function $F = F(x,y)$ where $y = y(x)$. Then F is an <u>implicit</u> function of x. Use chain and product rule to determine $\frac{dF}{dx}$. Called <u>implicit differentiation</u>

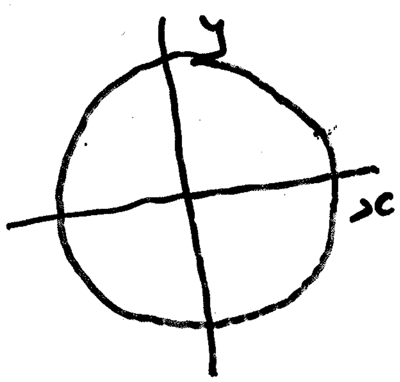
In general,
$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

Important for solving certain types of diff. equations

<u>Example</u>: Solve $y' = -\frac{x}{y}$

$$\Rightarrow \int y\,dy = -\int x\,dy \Rightarrow \tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + C$$

$$\Rightarrow x^2 + y^2 = 2C$$



Circle radius $\sqrt{2C}$ centred on origin!

Define $F(x,y) = x^2 + y^2$. Then

$$F(x,y) = 2C = \text{constant}$$

is solution to diff. equation. <u>Check</u> using implicit differentiation

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{\partial F}{\partial x} = 2x \quad , \quad \frac{\partial F}{\partial y} = 2y$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$ (what we started with)

<u>Question</u>: Can we generalize?

Yes! method to find implicit solution of certain type of diff. equation known as ....

Exact Differential Equations

Consider : $F(x,y) = C = \text{constant}$

Differentiate: $\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

Define :

$$P(x,y) = \frac{\partial F}{\partial x} \qquad Q(x,y) = \frac{\partial F}{\partial y}$$

$$\Rightarrow \qquad P(x,y) + Q(x,y)\frac{dy}{dx} = 0 \qquad \ldots (1)$$

But, $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x}$

$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial x \partial y}$

Hence, $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \ldots (2)$

Diff. equation of type (1) called *exact* if equation (2) satisfied.

General solution to exact diff. equation can <u>always</u> be written in implicit form
<u>$F(x,y) = \text{constant}$.</u>

<u>Example</u> : Show diff. equation $y' = -x/y$ is exact and find general solution.

$$x + y\frac{dy}{dx} = 0 \quad \Rightarrow \quad P(x,y) = x$$
$$Q(x,y) = y$$

$$\Rightarrow \quad \frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x} \qquad \text{exact !}$$

Now must determine function $F(x,y)$ that satisfies conditions

$$\frac{\partial F}{\partial x} = P(x,y) \qquad \ldots (3)$$

$$\frac{\partial F}{\partial y} = Q(x,y) \qquad \ldots (4)$$

First integrate (3) with respect to x :

$$F(x,y) = \int P(x,y)\,dx = \int x\,dx = \frac{1}{2}x^2 + C_1(y) \qquad \ldots (5)$$

Important : $C_1(y)$ is an arbitrary function of y. Determine by taking partial deriv. of Eq. (5) with respect to y:

$$\frac{\partial F}{\partial y} = \frac{dC_1}{dy}$$

But $\frac{\partial F}{\partial y} = Q(x,y) = y \Rightarrow \frac{dC_1}{dy} = y$

Interpret this condition as diff. equation where y is independent variable. Solve by direct integration :

$$\int dC_1 = C_1(y) = \int y\,dy = \frac{1}{2}y^2 + C$$

Substitute into equation (5) to get first integral :

$$F = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$$

Implicit solution given by

$$F = \frac{1}{2}x^2 + \frac{1}{2}y^2 = \text{constant.}$$

Method to find general solution to Exact Differential Equations

Applies to diff. equation that can be written as

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0 \qquad \dots (1)$$

1. First show (1) is exact. Exact if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

2. Then the general solution is given by

$$F(x,y) = \text{constant}$$

where function $F = F(x,y)$ defined by conditions

$$\frac{\partial F}{\partial x} = P(x,y) \qquad \dots (2)$$

$$\frac{\partial F}{\partial y} = Q(x,y) \qquad \dots (3)$$

3. Integrate equation (2) with respect to x (treat y as fixed constant):

$$\int \frac{\partial F}{\partial x}\,dx = F(x,y) = \int P(x,y)\,dx + C_1(y) \quad \ldots(4)$$

important: arbitrary function of y.

Instead of arbitrary integration constant, have arbitrary function $C_1(y)$ of y

4. Take partial derivative of (4) with respect to y and compare with (3):

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int P\,dx\right) + \frac{dC_1}{dy} = Q(x,y)$$

$$\Rightarrow \quad \frac{dC_1}{dy} = \underbrace{Q(x,y) - \frac{\partial}{\partial y}\left(\int P\,dx\right)} \quad \ldots(5)$$

* note, right-hand side will be independent of x. (Useful check have done maths right).

5. Call right-hand side of (5) $f = f(y)$

$$\Rightarrow \quad \frac{dC_1}{dy} = f(y)$$

6. Integrate to get $C_1(y)$

$$dC_1 = f(y)dy \Rightarrow \int dC_1 = C_1(y) = \int f(y)dy + \underset{\substack{\text{arbitrary} \\ \text{constant.}}}{C}$$

7. Subsitute back into equation (4) to get what is called the first integral.

Finally general solution (in implicit form) is

$$\int P_1(x,y)\,dx + C_1(y) = \text{constant}$$

<u>Example</u>: Show that the diff. equation

$$3x^2 + y - (3y^2 - x)\frac{dy}{dx} = 0$$

is exact and find general solution in implicit form.

$$P(x,y) = 3x^2+y \Rightarrow \frac{\partial P}{\partial y} = 1$$

$$Q(x,y) = -(3y^2-x) \Rightarrow \frac{\partial Q}{\partial x} = 1$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

so exact!

$$\frac{\partial F}{\partial x} = P \Rightarrow F(x,y) = \int P(x,y)\,dx$$

$$\Rightarrow F(x,y) = \int (3x^2+y)\,dx = x^3 + xy + C_1(y)$$

$$\Rightarrow \frac{\partial F}{\partial y} = x + \frac{dC_1}{dy} = Q(x,y) = x - 3y^2$$

x terms cancel

$$\Rightarrow \frac{dC_1}{dy} = -3y^2 \Rightarrow dC_1 = -3y^2\,dy$$

$$\Rightarrow \int dC_1 = C_1(y) = \int (-3y^2)\,dy = -y^3 + C$$

$$\Rightarrow F(x,y) = x^3 + xy - y^3 + C$$

$\Rightarrow$ general solution is $F(x,y) = \text{constant}$

$$\Rightarrow x^3 + xy - y^3 + C = \text{constant} = C_2$$

$$\Rightarrow \underline{x^3 + xy - y^3 = C_2 - C \equiv D}.$$

Remark : Can always check have correct answer by implicit differentiation of solution to rederive the original diff. equation. Treat y as function of x :

$$\frac{dF}{dx} = 3x^2 + \frac{d}{dx}(xy) - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3x^2 + y + x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3x^2 + y + (x - 3y^2)\frac{dy}{dx} = 0$$

What we started with !

NOTE : more examples and notes on integrating factors are on pages 14-18 of the course notes.