Example 1:

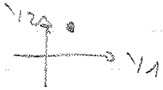
$$\frac{dy_1}{dx} = (-2 - y_1 + y_2)\, y_1 \qquad \frac{dy_2}{dx} = (1 - y_1)\, y_2$$

fixed points: $(-2-y_1+y_2)y_1 = 0$ and $(1-y_1)y_2 = 0$

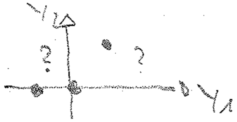
- $y_1^{Fix} = 0,\ y_2^{Fix} = 0$

- $-2-y_1+y_2 = 0,\ y_2 = 0 \Rightarrow y_1^{Fix} = -2,\ y_2^{Fix} = 0$

- $-2-y_1+y_2 = 0,\ 1-y_1 = 0 \Rightarrow y_1^{Fix} = 1,\ y_2^{Fix} = 3$



(partial) phase portrait



How does the x-dependent solution in the vicinity of the fixed point look like → "Linear stability"

Idea: Use $y_1(x) = y_1^{Fix} + \eta_1(x) \quad y_2(x) = y_2^{Fix} + \eta_2(x)$

with $\eta_1(x)$ and $\eta_2(x)$ being "small" and expand the right hand sides ("Taylor expansion")

Example 2: $y_1' = (-2-y_1+y_2)y_1,\ y_2' = (1-y_1)y_2$

$y_1^{Fix} = 0,\ y_2^{Fix} = 0$: $\quad y_1(x) = y_1^{Fix} + \eta_1(x) = \eta_1(x)$

$y_2(x) = y_2^{Fix} + \eta_2(x) = \eta_2(x)$

$$\Rightarrow \frac{dy_1}{dx} = \frac{d\eta_1}{dx} = (-2-\eta_1+\eta_2)\eta_1 = -2\eta_1 - \eta_1^2 + \eta_1\eta_2$$

$$\frac{dy_2}{dx} = \frac{d\eta_2}{dx} = (1-\eta_1)\eta_2 = \eta_2 - \eta_1\eta_2$$

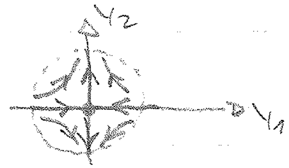
Approximation: "$\eta^2 \ll \eta \ll 1$"

$$\frac{d\eta_1}{dx} = -2\eta_1 + \cancel{\ldots} \qquad \frac{d\eta_2}{dx} = \eta_2 + \cancel{\ldots} \qquad \text{Linear system!}$$

$$\underline{\underline{A}} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow p(\lambda) = (-2-\lambda)(1-\lambda) \Rightarrow \lambda^{(A)} = -2,\ \lambda^{(B)} = 1$$

$\Rightarrow$ Saddle

phase portrait (local)



"unstable fixed point"

Example 3: $y_1' = (-2-y_1+y_2)y_1,\ y_2' = (1-y_1)y_2$

$y_1^{Fix} = -2,\ y_2^{Fix} = 0$: $\quad y_1(x) = y_1^{Fix} + \eta_1(x) = -2 + \eta_1(x)$

$y_2(x) = y_1^{Fix} + \eta_2(x) = \eta_2(x)$

$$\frac{dy_1}{dx} = \frac{d\eta_1}{dx} = (-\cancel{2} - (-\cancel{2} + \eta_1) + \eta_2)(-2+\eta_1)$$

$$= 2\eta_1 - 2\eta_2 - \eta_1^2 + \eta_1\eta_2$$

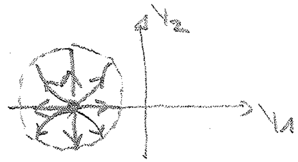
$$\frac{dy_2}{dx} = \frac{d\eta_2}{dx} = (1-(-2+\eta_1))\eta_2 = 3\eta_2 - \eta_1\eta_2$$

Linearisation:

$$\frac{d\eta_1}{dx} = 2\eta_1 - 2\eta_2 + \cancel{\ldots} \qquad \underline{\underline{A}} = \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix} \qquad \lambda^{(A)} = 2,\ \lambda^{(B)} = 3 \Rightarrow \text{unstable node}$$

$$\frac{d\eta_2}{dx} = 3\eta_2 + \cancel{\ldots}$$

phase portrait (local)



"unstable fixed point"

Example 4: $y_1' = (-2 - y_1 + y_2)\, y_1 \,,\; y_2 = (1 - y_1)\, y_2$

$y_1^{Fix} = 1 \,,\; y_2^{Fix} = 3 \qquad y_1(x) = y_1^{Fix} + \eta_1(x) = 1 + \eta_1(x)$

$$y_2(x) = y_2^{Fix} + \eta_2(x) = 3 + \eta_2(x)$$

$$\frac{dy_1}{dx} = \frac{d\eta_1}{dx} = (-2 - 1 - \eta_1 + 3 + \eta_2)(1 + \eta_1) = -\eta_1 + \eta_2 - \eta_1^2 + \eta_1 \eta_2$$

$$\frac{dy_2}{dx} = \frac{d\eta_2}{dx} = (1 - 1 - \eta_1)(3 + \eta_2) = -3\eta_1 - \eta_1 \eta_2$$

Linearisation

$$\frac{d\eta_1}{dx} = -\eta_1 + \eta_2 + \cancel{\dots}$$

$$\frac{d\eta_2}{dx} = -3\eta_1 + \cancel{\dots}$$

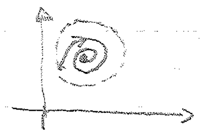
$$\underline{\underline{A}} = \begin{pmatrix} -1 & 1 \\ -3 & 0 \end{pmatrix}$$

$$\Rightarrow p(\lambda) = (-1-\lambda)(-\lambda) + 3 = \lambda^2 + \lambda + 3 = (\lambda + 1/2)^2 + \frac{11}{4}$$

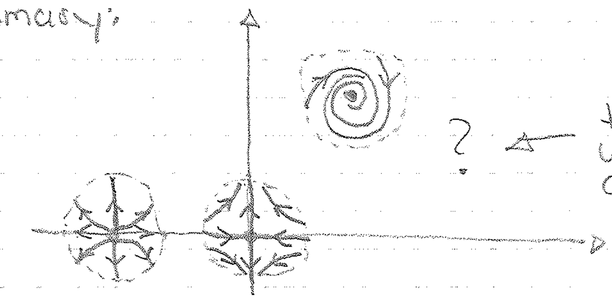
$$\Rightarrow \lambda^{(A)} = -1/2 - i\frac{\sqrt{11}}{2} \,,\; \lambda^{(B)} = -1/2 + i\frac{\sqrt{11}}{2}$$

(stable focus)

phase portrait
(local)



Summary:



? ← the undiscovered country

to be continued ...

— End —