Queen Mary
University of London

MAS/118 Differential Equations
Marking scheme

Problem 1 (similar to coursework)

a)

$$\underbrace{y = \int \frac{3x}{(1-x)(2+x)}\,dx}_{\boxed{2P}} = \int \underbrace{\frac{1}{1-x} - \frac{2}{2+x}}_{\boxed{2P}}\,dx = \underbrace{-\ln|1-x|}_{\boxed{2P}} - \underbrace{2\ln|2+x|}_{\boxed{2P}} + \underbrace{C}_{\boxed{2P}}$$

b)

$$\begin{aligned} e^{-y}dy &= e^x dx \quad \boxed{2P} \\ -e^{-y} &= e^x + C \quad \boxed{2P} \\ -1 &= 1 + C \quad \Rightarrow \quad C = -2 \quad \boxed{2P} \\ y &= -\ln(2 - e^x) \quad \boxed{2P} \end{aligned}$$

c)

$$\begin{aligned} y' &= \frac{1}{\sinh(x)} - \frac{x}{\sinh^2(x)} \underbrace{\cosh x}_{\sqrt{1+\sinh^2(x)}\ \boxed{2P}} \quad \boxed{2P} \\ &= \frac{y}{x} - \frac{y^2}{x}\sqrt{1+(x/y)^2} \quad \boxed{2P} \\ &= \frac{y}{x}(1 - \sqrt{y^2+x^2}) \quad \boxed{2P} \\ y(0) &= \lim_{x\to 0} \frac{x}{\sinh(x)} \quad \boxed{2P} \\ &= \lim_{x\to 0} \frac{1}{\cosh(x)} = 1 \quad \boxed{2P} \end{aligned}$$

Problem 2 (similar to coursework)

a)

$$\underbrace{2(1+x) - y^2/(1+x)}_{P} + \underbrace{2(1+x+y)}_{Q}\, y' = 0$$

$$\frac{\partial P}{\partial y} = -2y/(1+x) \quad \boxed{1P}, \ \frac{\partial Q}{\partial x} = 2 \quad \boxed{1P} \quad \Rightarrow \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \boxed{2P}$$

b)

$$2-\frac{y^2}{(1+x)^2}+2\left(1+\frac{y}{1+x}\right)y'=0 \quad \boxed{1P}$$

$$\frac{\partial F}{\partial x}=2-\frac{y^2}{(1+x)^2} \quad \Rightarrow \quad F=\underbrace{2x+\frac{y^2}{1+x}}_{\boxed{1P}}+\underbrace{K(y)}_{\boxed{2P}}$$

$$\frac{\partial F}{\partial y}=2\left(1+\frac{y}{1+x}\right)=\frac{2y}{1+x}+K'(y) \quad \boxed{2P} \quad \Rightarrow K(y)=2y(+C) \quad \boxed{1P}$$

$$2x+\frac{y^2}{1+x}+2y=C \quad \boxed{1P}$$

c)

$$C=0+4-4=0 \quad \boxed{1P}$$

$$2x+\frac{y^2}{1+x}+2y=0 \quad \Rightarrow \quad (y+1+x)^2=(1+x)^2-2x(1+x)=1-x^2 \quad \boxed{1P}$$

$$y=-(1+x)\underbrace{-}_{\boxed{4P}}\sqrt{1-x^2} \quad \boxed{2P}$$

Problem 3 (similar to coursework)

a)

$$\lambda^2+3\lambda+2=0 \quad \boxed{1P} \quad \Rightarrow \quad \lambda^{(A)}=-1, \quad \lambda^{(B)}=-2 \quad \boxed{2P}$$

$$y=C^{(A)}e^{-x}+C^{(B)}e^{-2x} \quad \boxed{1P}$$

b)

$$y_P=A(x)e^{-x}+B(x)e^{-2x} \quad \boxed{1P}$$

$$\begin{aligned} A(x) &= \int \frac{e^x/(1+e^x)^2}{-1+2}dx=-\frac{1}{1+e^x} \quad \boxed{2P} \\ B(x) &= \int \frac{e^{2x}/(1+e^x)^2}{-2+1}dx=-\frac{1}{1+e^x}-\ln(1+e^x) \quad \boxed{2P} \end{aligned}$$

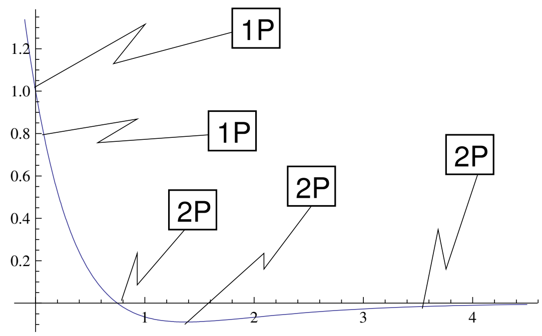
$$y_P=-\frac{e^{-x}}{1+e^x}-\frac{e^{-2x}}{1+e^x}-e^{-2x}\ln(1+e^x)=-e^{-2x}-e^{-2x}\ln(1+e^x) \quad \boxed{3P}$$

$$y=C^{(A)}e^{-x}+C^{(B)}e^{-2x}-e^{-2x}\ln(1+e^x) \quad \boxed{2P}$$

c)

$$\begin{aligned} y(0) &= C^{(A)} + C^{(B)} - \ln 2 = 1 \quad \boxed{1\text{P}} \\ y'(0) &= -C^{(A)} - 2C^{(B)} + 2\ln 2 - 1/2 = -3 \quad \boxed{1\text{P}} \end{aligned}$$

$$C^{(A)} = -1/2, \quad C^{(B)} = 3/2 + \ln 2 \quad \boxed{2\text{P}}$$



Problem 4 (similar to coursework)

a)

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 2/3 \\ -4/3 & -1/3 \end{pmatrix} \Rightarrow 0 = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \underbrace{\lambda^2 - 2\lambda/3 + 5/9}_{\boxed{2\text{P}}} = (\lambda + 1/3)^2 + 4/9$$

$$\lambda^{(A)} = 1/3 + 2i/3 \quad \boxed{1\text{P}}, \quad \lambda^{(B)} = 1/3 - 2i/3 \quad \boxed{1\text{P}}$$

$$\begin{pmatrix} 2/3 - 2i/3 & 2/3 \\ -4/3 & -2/3 - 2i/3 \end{pmatrix} \underline{u}^{(A)} = 0 \quad \Rightarrow \quad \underline{u}^{(A)} = \begin{pmatrix} 1 \\ -1 + i \end{pmatrix} \quad \boxed{2\text{P}}$$

$$\begin{pmatrix} 2/3 + 2i/3 & 2/3 \\ -4/3 & -2/3 + 2i/3 \end{pmatrix} \underline{u}^{(B)} = 0 \quad \Rightarrow \quad \underline{u}^{(B)} = \begin{pmatrix} 1 \\ -1 - i \end{pmatrix} \quad \boxed{2\text{P}}$$

b)

$$\underline{y}(x) = C^{(A)} \begin{pmatrix} 1 \\ -1 + i \end{pmatrix} e^{(1/3+2i/3)x} + C^{(B)} \begin{pmatrix} 1 \\ -1 - i \end{pmatrix} e^{(1/3-2i/3)x} \quad \boxed{2\text{P}}$$

c)

$$\begin{aligned} y_1(0) &= 1 = C^{(A)} + C^{(B)} \quad \boxed{1\text{P}} \\ y_2(0) &= -1 = -(C^{(A)} + C^{(B)}) + i(C^{(A)} - C^{(B)}) \quad \boxed{1\text{P}} \end{aligned}$$

$$C^{(A)} = 1/2, \quad C^{(B)} = 1/2 \quad \boxed{2\text{P}}$$

$$\begin{aligned} y_1(x) &= \frac{1}{2} e^{x/3} \left(e^{2ix/3} + e^{-2ix/3} \right) \\ &= e^{x/3} \cos(2x/3) \quad \boxed{2\text{P}} \\ y_2(x) &= -\frac{1}{2} e^{x/3} \left(e^{2ix/3} + e^{-2ix/3} \right) + \frac{i}{2} e^{x/3} \left(e^{2ix/3} - e^{-2ix/3} \right) \\ &= -e^{x/3} \cos(2x/3) - e^{x/3} \sin(2x/3) \quad \boxed{2\text{P}} \end{aligned}$$

d) F $\boxed{6\text{P}}$ (D $\boxed{3\text{P}}$, C,E $\boxed{1\text{P}}$)