Isolated systems in General Relativity

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1. General Relativity

- Einstein's theory of General Relativity is the best theory of gravitation we have. It describes the gravitational field as a manifestation of the curvature of the spacetime fabric. The equations governing the gravitational field, the *Einstein field equations*, are exceedingly complicated. In particular, the equations are highly nonlinear. This feature implies that the gravitational field can be its own source. The nonlinear character of the gravitational field is responsible for some of its more intriguing predictions, like that of the existence of black holes and the generation of gravitational waves.

Box 1: the Einstein equations. These are second order partial differential equations for the metric tensor $g_{\mu\nu}$ —sometimes also called the gravitational potential. They have the form:

$$g^{\alpha\beta}\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha x^\beta}+\frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu x^\nu}-\frac{\partial^2 g_{\mu\beta}}{\partial x^\alpha x^\nu}-\frac{\partial^2 g_{\alpha\nu}}{\partial x^\mu x^\beta}\right)+H_{\mu\nu}(g,\partial g)=0.$$

The nonlinearity of the equations is contained in the term $H_{\mu\nu}(g,\partial g)$. An easy heuristic argument can be used to show that it contains terms of order g^8 and $(\partial g)^2$. Associated to the field equations there is a certain freedom in the choice of coordinates (gauge choice) which may be used to bring the equations into some tractable form.

2. Isolated systems and asymptotics

- This research project is concerned with the general relativistic description of the so-called *isolated systems* —like black holes and neutron stars. It exploits the fact that regardless how complicated a system containing a black hole or a star may be, if looked from far away, the system may appear simple —in the same way that a person looked from far away looks almost like a point, with most of its individual features faded away. This is the main idea of what is known as the *theory of the asymptotics of the gravitational field*.

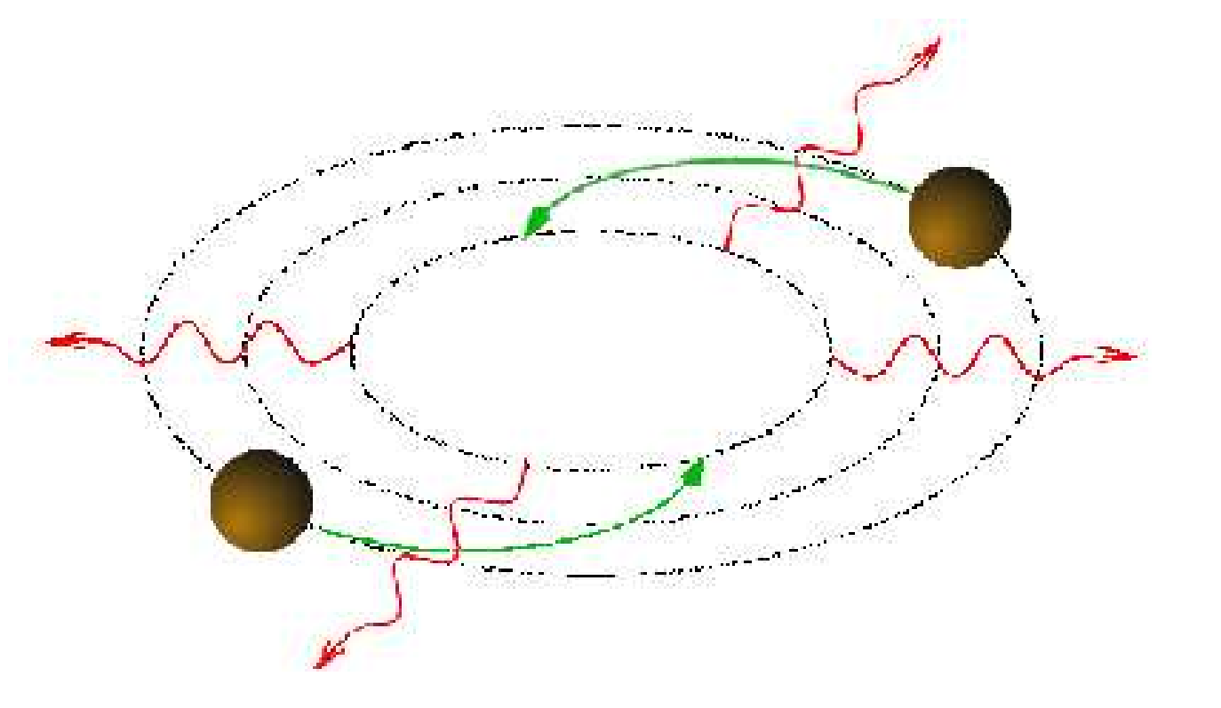

FIGURE 1. Prototype of an isolated system: two black holes spinning around each other. Eventually they will coalesce to form a bigger black hole. In the process gravitational radiation will be emitted —accounting up to perhaps 10% of the mass of the system.

- By means of an asymptotic approach the equations simplify enough so that a precise mathematical description of, say, gravitational waves and black holes is possible. Relativistic phenomena produced by systems which can be considered in a very good approximation as isolated systems are starting to be detected. Hence it is imperative to have a rigorous framework for their satisfactory discussion.
- A natural and in some senses fundamental physical question is the following:

How does the gravitational field of an isolated system look like far away from the sources?

- At more detailed level the latter question can be rephrased as: which are the appropriate boundary (asymptotic) conditions to describe an isolated system in General Relativity? An attractive framework for the discussion of these issues was suggested, some 40 years ago, by Sir Roger Penrose. His idea consists in attaching a boundary to the spacetime by means of a mathematical construction known as *conformal compactification*.

Box 2: conformal compactification. It is a mathematical technique to "bring" infinity to a finite distance. This is done by attaching a point or a series of points —which will represent infinity— to the manifold under study in such a way that angles are preserved in the resulting "compactified manifold".The classical example is the compactification of the Euclidean plane into the sphere by means of stereographic coordinates. In the case of the empty spacetime (Minkowski spacetime), its conformally compactified version is shown below.

- Penrose's crucial insight was to suggest that the conformal compactification of a spacetime describing an isolated system should have the same structure as the conformal compactification of the empty spacetime (*Minkowski spacetime*) —see box 2 and figure 2. In Minkowski spacetime one can escape to infinity along the paths followed by light rays ($\mathscr{I}^+$ and $\mathscr{I}^-$, *future and past null infinity*), along curves with a speed lower than that of light (i^+ and i^-, *future and past timelike infinity*), and along curves with a speed higher than that of light (i^0, *spatial infinity*).

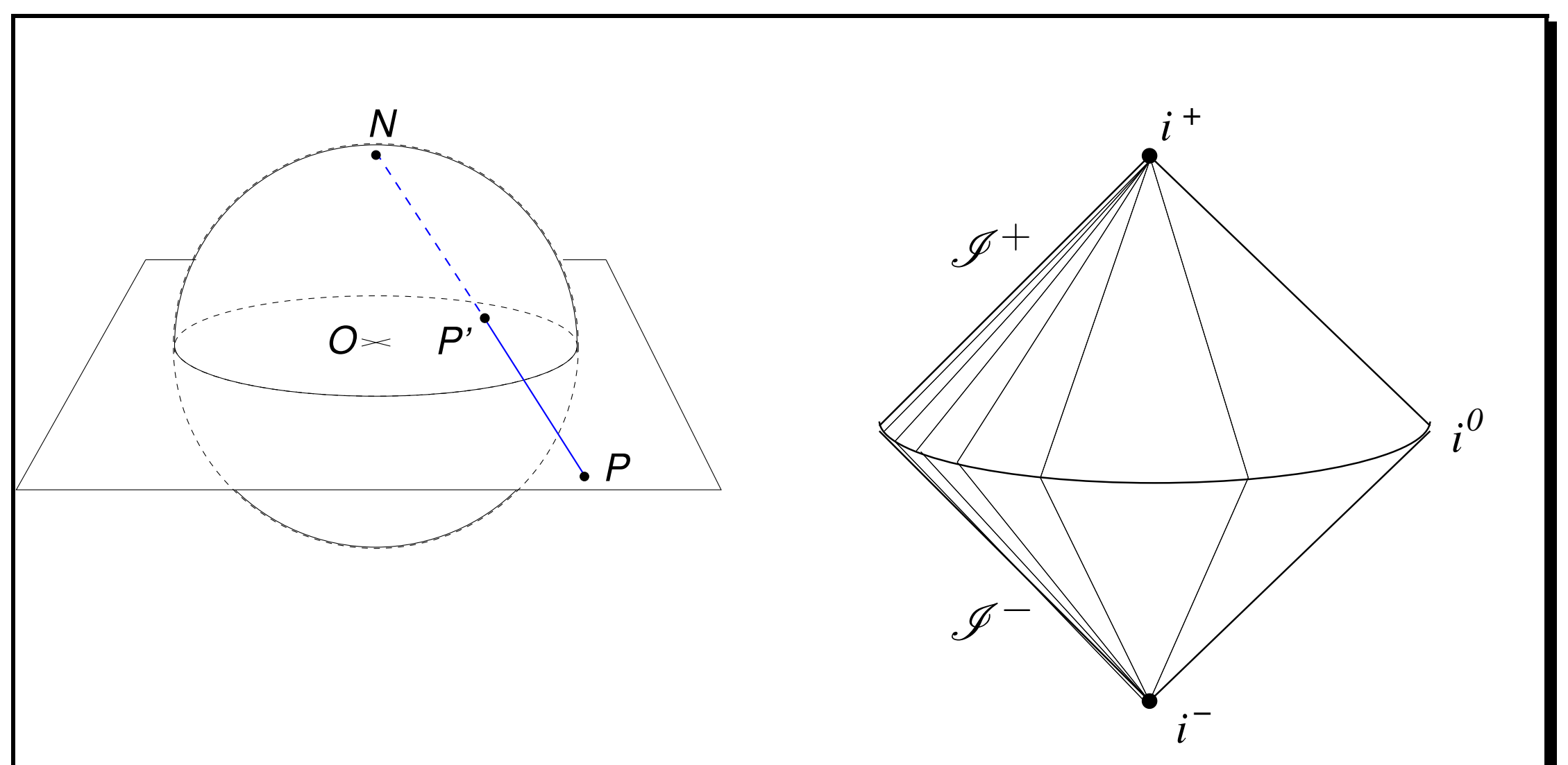

FIGURE 2. LEFT. Conformal compactification of the Euclidean plane. The point at "infinity" in the plane corresponds to the North Pole of the sphere. **RIGHT.** Conformal compactification of the Minkowski spacetime. The sets $\mathscr{I}^+$, $\mathscr{I}^-$, i^0, i^+, i^- correspond to the different ways one can escape to infinity.

Penrose's proposal: spacetimes describing isolated systems should admit a conformal compactification similar to that of Minkowski spacetime.

- Among the advantages of this approach to the study of isolated systems in General Relativity one has to mention: it provides a *geometrical setting* which is very much in the general spirit of the theory; it allows to replace dubious limiting procedures in asymptotic expansions by rigorous local *differential geometric calculations*; it provides a framework on which diverse notions of physical interest can be uniquely and rigorously defined and manipulated —-gravitational radiation, mass loss, polarisation states; it provides a platform for the discussion of global issues —like the notion of black holes, cosmic censorship— and a starting point for the numerical computation of complete spacetimes.

3. Constructing the spacetimes

- It is *a priori* not clear at all whether the class spacetimes satisfying Penrose's is big enough to cover systems of astrophysical interests. From a mathematical point of view, the proposal postulates the existence of a class of solutions to the Einstein field equations with a very detailed behaviour at null infinity ($\mathscr{I}^+$ and $\mathscr{I}^-$). Particular examples of exact solutions to the Einstein field equations —that is, known in closed form— satisfy the asymptotic conditions suggested by Penrose. However, almost all of these examples are *stationary* —that is, they contain no gravitational radiation!

Key questions:

(i) Are there any solutions to the Einstein equations containing gravitational radiation, and that satisfy the asymptotic behaviour suggested by Penrose?
(ii) How big is this class of solutions?
(iii) How can one construct these solutions?
(iv) What kind of systems cannot be described by these ideas?

- There are essentially two approaches to address the above questions: the use of *exact solutions* of the Einstein field equations; and the construction by means of diverse mathematical techniques of *non-explicit (abstract) solutions*, which hence are shown to exist.

Box 3: exact solutions versus abstract solutions.
Exact solutions. Here one makes use of diverse simplifying assumptions —like the existence of certain number symmetries (both discrete and continuous) in the solutions, or a certain algebraic behaviour in the tensors describing the curvature of the spacetime. Thus one obtains certain ordinary differential equations whose solutions are known. The exact solutions approach is of great help for the construction of specific examples and for gaining intuition into the nonlinear aspects of General Relativity. On the other hand, it restricts the analysis to the class of spacetimes that has been constructed, so it is not clear if certain features of the solutions can be extended to more general families of solutions.
Abstract solutions. Here one makes use of the machinery of the theory of partial differential equations and functional analysis to prove that solutions to the Einstein equations with certain properties exist (or not) and possess (or not) certain desirable properties. Due to the abstract (e.g. non-explicit) nature of the analysis it is harder to discuss particular features of the individual solutions. Even in this approach one has to make certain assumptions to make the mathematical argumentation go through —usually a more restricted assumption on the symmetries of the solutions, or that the solutions are close (in a precise mathematical sense) to a certain exact solution.

- In an attempt to understand Penrose's proposal and answering the questions raised, my research decants for a strategy mainly based in the analysis of abstract solutions —that is, I analyse the mathematical existence of solutions to the Einstein equations corresponding to isolated systems. A starting point for this approach is to make use of the *initial value problem* for the Einstein field equations. That is, one prescribes the initial geometry —*initial data*—of the spacetime at a certain instant of time and then one attempts to reconstruct the rest of the spacetime lying at the future (or past!) of the initial instant.
- Closely tied to the question of the mathematical existence of solutions is the study of their global properties —how do the solutions behave at the different infinities $\mathscr{I}^+$, $\mathscr{I}^-$, i^+, i^-, i^0? Are there singularities in the solutions? Are there black holes forming?
- The qualitative analysis of the solutions to the Einstein equations is not a luxury or a theoretical eccentricity. It is a desirable prerequisite for further studies, like numerical simulations, from where a direct contact with astronomical observations is possible. This type of research is an endeavour on which I make use of sophisticated, state of the art techniques in a wide array of mathematical disciplines —differential geometry, differential equations, functional analysis, and computer algebra— among others to make rigorous assertions about the existence, uniqueness asymptotic behaviour and global properties of the solutions to the equations governing the gravitational field. It is until very recently that the mathematical techniques necessary to handle the Einstein equations in its asymptotic aspects have been developed. However, the nature of the equations and the particular needs of my studies are such that one can rarely make use of off-the-shelf techniques. Hence, I have very often to tailor my own mathematical methods to the particular aspects of the analysis under question.

4. A rigidity conjecture

- As a result of my investigations, I have been able to produce a very precise description of what should be the asymptotic form of the solutions to the Einstein field equations which describe isolated systems. The latter is a necessary first step in order to be able to produce a general existence statement about the solutions under scrutiny —this being a long term objective. My analysis has led to detailed conjectures about the applicability and generality of Penroses ideas about the description of isolated systems in General Relativity. As an example of the statements one would like to prove in a rigorous manner we have the following,

Conjecture (Valiente Kroon, 2004,2005). *The initial data giving rise to a spacetime satisfying Penrose's proposal has to be stationary close to spatial infinity i^0.*

- If proved correct, the above result constitutes a rigidity result. That is, the requirement of having a solution to the Einstein equations with specific properties imposes severe conditions on the initial geometry giving rise to the solution.

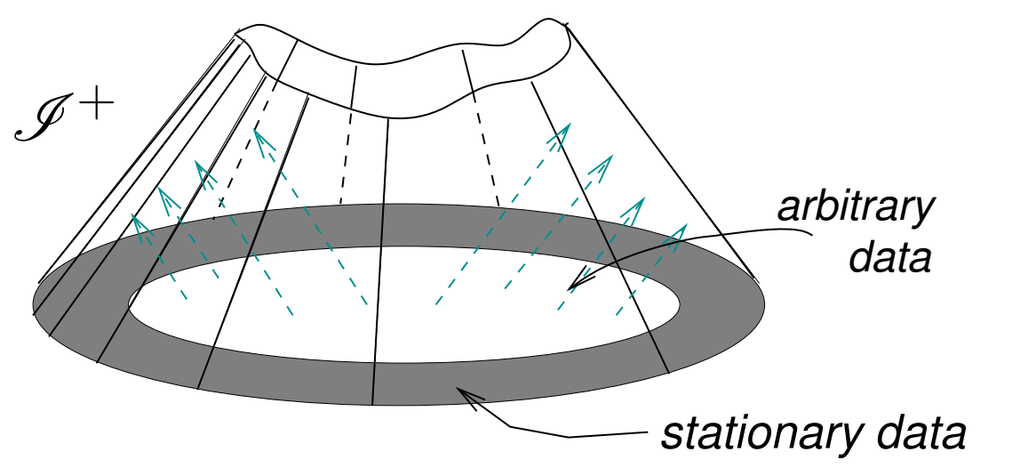

FIGURE 3. Schematic depiction of the conjecture. The initial data has to be stationary near spatial infinity, but can be arbitrary inside a compact set. In the region of $\mathscr{I}^+$ close to i^0 there is no radiation, however after a while the radiation contained in the nonstationary portion of the initial geometry registers in null infinity according to Penrose's prescription.

- My current research is geared into making this kind of conjectures precise and if possible transforming them into rigorous statements —*theorems*.

5. Social Relevance

Mathematics is the natural language of Physics and in great extent also that of technology. The formalisation of a physical theory requires the development of novel mathematical techniques which can be later employed in areas which are sometimes completely disconnected from their original domain of application. Setting in firm and rigorous ground the basis of a theory enables the researchers to concentrate in the physics of the problem. At a different level, it should be said that the understanding of Nature by means of Mathematics is a central aspect of our culture.

6. Acknowledgements

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7. References

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