Initial data sets for the Schwarzschild spacetime

—in collaboration with A. García-Parrado—

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Question:

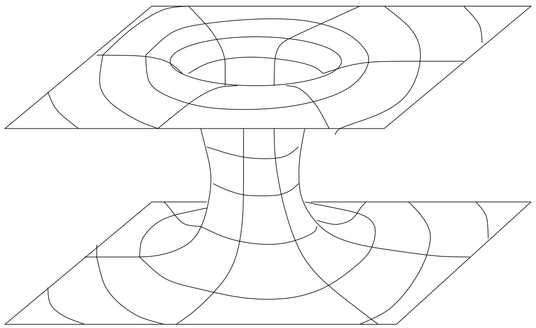
given an initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ satisfying the Einstein constraint equations

$$r + K - K^{ij}K_{ij} = 0,$$
$$D^j K_{ij} - D_i K = 0,$$

under which conditions it corresponds to an hypersurface of the Schwarzschild spacetime?

Example:

- The standard $t = constant$ slice in isotropic coordinates



$$h_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij},$$
$$K_{ij} = 0.$$

Strategy:

Need a characterisation of the Schwarzschild spacetime that is amenable to a 3+1 decomposition —in the spirit of the Cauchy problem for the Einstein vacuum equations.

[a] *An intrinsic characterization of the Schwarzschild spacetime*, Class. Quantum Grav. **15**, 1323 (1998)

Strategy:

Need a characterisation of the Schwarzschild spacetime that is amenable to a 3+1 decomposition —in the spirit of the Cauchy problem for the Einstein vacuum equations.

- Ferrando & Sáez [a] have given a characterisation in terms of concomitants of the Weyl tensor, $C_{\mu\nu\lambda\rho}$, which is appropiate for our purposes.

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One can obtain a 3+1 decomposition of the aforementioned characterisation writing the Weyl tensor

$$C_{\mu\nu\lambda\sigma} = 2\left(l_{\mu[\lambda}E_{\sigma]\nu} - l_{\nu[\lambda}E_{\sigma]\mu} - n_{[\lambda}B_{\sigma]\tau}\epsilon^{\tau}{}_{\mu\nu} - n_{[\mu}B_{\nu]\tau}\epsilon^{\tau}{}_{\lambda\sigma} \right)$$

in terms of its electric, $E_{\mu\nu}$, and magnetic $B_{\mu\nu}$ parts:

$$E_{\tau\sigma} \equiv C_{\tau\nu\sigma\lambda}n^{\nu}n^{\lambda}, \quad B_{\tau\sigma} \equiv C^{*}_{\tau\nu\sigma\lambda}n^{\nu}n^{\lambda}.$$

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Note that by virtue of the Gauss-Codazzi equations, the above quantities can be determined on $\mathcal{S}$ purely in terms of the initial data, via:

$$E_{ij} = r_{ij} + KK_{ij} - K_{ik}K^{k}{}_{j}$$

$$B_{ij} = \epsilon^{kl}_{(i}D_{|k}K_{l|j)}.$$

Hence one obtains a set of necessary conditions for an initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ to be Schwarzschildean:

Proposition. *Let*

$$\rho = \left(\frac{1}{2}B_i{}^j B^{il} E_{jl} - \frac{1}{6}E_{ij}E_l{}^j E^{il}\right)^{1/3},$$

$$P = -\frac{1}{2}E^{ij}K_{ij} - \rho K - \frac{1}{6\rho}\epsilon^{jk}{}_i\left(E^{il}D_k B_{lj} + B^{il}D_k E_{lj}\right),$$

$$P_i = \frac{1}{6\rho}(-B^{kl}D_i B_{kl} + E^{kl}D_i E_{kl}).$$

Necessary conditions for an initial data set (h_{ij}, K_{ij}), satisfying the Einstein vacuum constraints, on a manifold $\mathcal{S}$ to be a Schwarzschild initial data set are:

$$B_{ij} = -\frac{1}{\rho}(B_i{}^k E_{kj} + B_j{}^k E_{ki}) \qquad E_{ij} = \frac{1}{\rho}(B_i{}^k B_{kj} - E_i{}^k E_{kj}) + 2\rho h_{ij},$$

$$B_{ij}P^i P^j = 0, \qquad E_{jk}\epsilon^k{}_{li}P^j P^l - P B_{ij}P^j = 0,$$

$$(P^2 + P^k P_k)B_{ij} + 2P E_{k(i}\epsilon^k{}_{j)l}P^l - 2P_{(i}B_{j)l}P^l = 0,$$

$$\frac{1}{3}P_k P^k - P^2 + \frac{2}{3}E_{kl}P^k P^l > 0, \qquad \frac{1}{9\rho^2}(P^k P_k - P^2) + 2\rho > 0.$$

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- Alternative, note that the characterisation by F & S yields a formula for the timelike Killing vector of the Schwarzschild spacetime in terms, again, of concomitants of $C_{\mu\nu\lambda\rho}$.

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- A KID is a pair (Y, Y^i) defined on $\mathcal{S}$ satisfying the equations

$$D_{(i}Y_{j)} - YK_{ij} = 0,$$
$$D_iD_jY - \mathcal{L}_{Y^l}K_{ij} = Y(r_{ij} + KK_{ij} - 2K_{il}K^l{}_j).$$

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- Given $(\mathcal{S}, h_{ij}, K_{ij})$ possessing a KID, then the development of the initial data has a Killing vector whose pull back to $\mathcal{S}$ coincides with the KID.

The KID candidate:

$$\Upsilon = \frac{\sqrt{|\rho P_i P^i - E_{ij} P^i P^j|}}{\rho^{11/6}\sqrt{3}},$$

$$\Upsilon^i = \frac{-\rho P P^i + E^{ij} P_j P - B^{kl} P_l P_m \epsilon^{im}{}_k}{\rho^{11/6}\sqrt{3|\rho P_j P^j - E_{kl} P^k P^l|}}.$$

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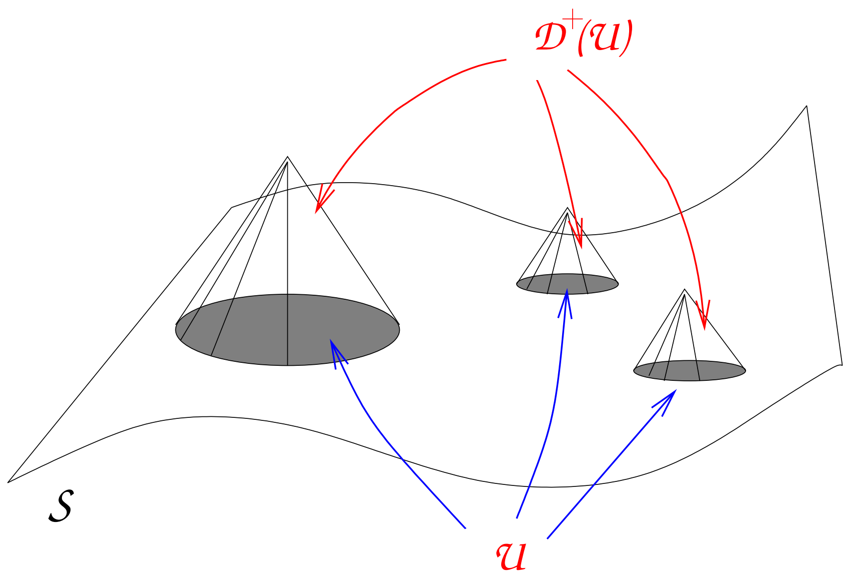
- Proven for the time symmetric case ($K_{ij} = 0$).

- If one imposes that the initial data contains a KID of the given form one can ensure that the necessary conditions are propagated in the domain of dependence $\mathcal{D}^+(\mathcal{U})$ of the parts of $\mathcal{S} \supset \mathcal{U}$ where the KID is timelike ($Y^2 - Y^i Y_i < 0$).



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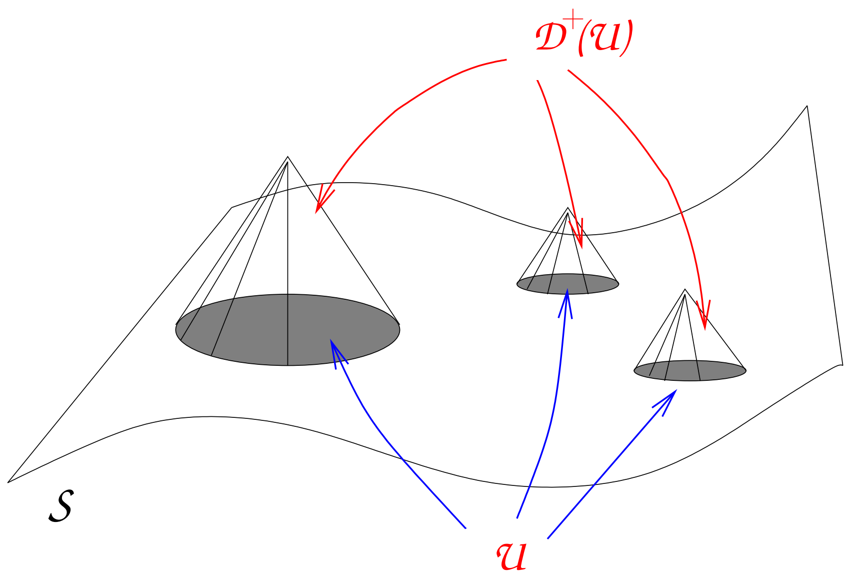

- In this way one obtains a set of sufficient conditions.

Some references:

- J. A. Valiente Kroon, *Characterization of Schwarzschildean initial data*, Phys. Rev. D **72**, 084003 (2004). Also at `gr-qc/0504003`.
- A. García-Parrado & J. A. Valiente Kroon, *Initial data sets for the Schwarzschild spacetime*. In preparation.