

Understanding the structure of spatial infinity: a rigidity result

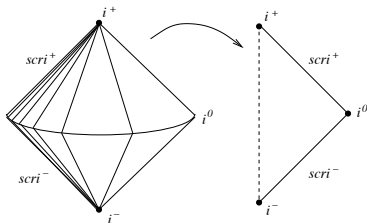
Juan A. Valiente Kroon

School of Mathematical Sciences
Queen Mary, University of London

j.a.valiente-kroon@qmul.ac.uk,

Space, Time and beyond
AEI, October 2009

Asymptotically simple spacetimes

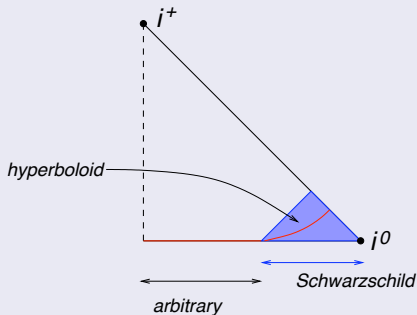


Definition (Penrose, 1963)

A smooth (C^∞) spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ is called asymptotically simple if there is another manifold $(\mathcal{M}, g_{\mu\nu})$ such that:

- (i) $\tilde{\mathcal{M}}$ is an open submanifold of \mathcal{M} with smooth boundary $\partial\tilde{\mathcal{M}} = \mathcal{I}$;
- (ii) there is a Ω on \mathcal{M} such that $g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$ on $\tilde{\mathcal{M}}$ and so that $\Omega = 0$, $d\Omega \neq 0$ on \mathcal{I} ;
- (iii) (every null geodesic in $\tilde{\mathcal{M}}$ acquires a future and a past endpoint on \mathcal{I});
- (iv) $\tilde{R}_{\mu\nu} = 0$.

A large class of asymptotically simple spacetimes?



- A large family of spacetimes that are asymptotically simple has been shown to exist (Chruściel & Delay, 2001):
 - initial data which is asymptotically Schwarzschildian near infinity —constructed using a refinement of the Corvino-Schoen gluing construction (2000);
 - semiglobal existence results for small (close to Minkowski) hyperboloidal data by H. Friedrich (1986).

More general asymptotically simple spacetimes?

General strategy:

- Discuss the existence of more general classes of asymptotically simple spacetimes using the Cauchy problem for the Einstein field equations.
 - In the spirit of the notion of asymptotic simplicity, work with quantities defined in the conformally rescaled spacetime $(\mathcal{M}, g_{\mu\nu})$.

More general asymptotically simple spacetimes?

General strategy:

- Discuss the existence of more general classes of asymptotically simple spacetimes using the Cauchy problem for the Einstein field equations.
 - In the spirit of the notion of asymptotic simplicity, work with quantities defined in the conformally rescaled spacetime $(\mathcal{M}, g_{\mu\nu})$.

Implementation:

- The Conformal Field Equations provide a suitable set of equations and unknowns. The equations are regular up to the points where the conformal factor vanishes (H. Friedrich, 1981).
- Initial data is provided on the point compactification of an asymptotically Euclidean surface.
 - The point at infinity of the data becomes the spatial infinity of the development.

The i^0 problem (continued)

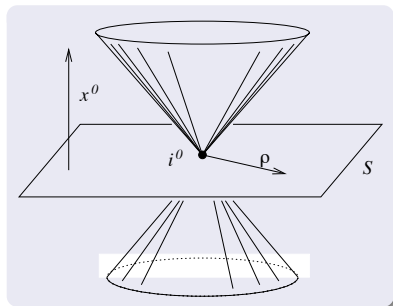
More precisely:

The point at infinity on the initial hypersurface is characterised by the conditions:

$$\Omega(i) = 0,$$

$$D_\alpha \Omega(i) = 0,$$

$$D_\alpha D_\beta \Omega(i) = -2h_{\alpha\beta}(i).$$

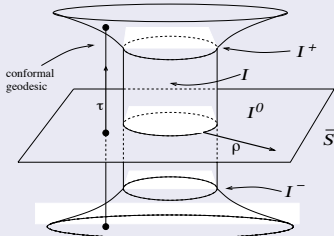


Singular behaviour of the rescaled Weyl tensor:

$$d^\alpha{}_{\beta\gamma\delta} = O\left(\frac{1}{|x|^3}\right) \quad \text{as } |x| \rightarrow 0$$

unless $m = 0$.

The cylinder at spatial infinity



Regular finite initial value problem at spatial infinity (H.Friedrich, 1998):

- The equations, field variables and data are regular.
- The location of the conformal boundary is known *a priori* and can be read from the initial data.

On the construction of the cylinder at spatial infinity

Blow up of i :

- the initial manifold \mathcal{S} is replaced by a manifold with boundary $\bar{\mathcal{S}}$ by blowing up i to a 2-sphere.

Conformal geodesics:

- the use of conformal Gaussian coordinates provides a canonical conformal factor, Θ , which can be calculated from the solutions to the constraint equations. The location of the conformal boundary is known *a priori*!

Rescaling of the spinor dyad:

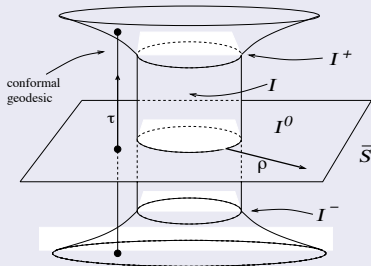
$$\delta_A \mapsto \kappa^{1/2} \delta_A,$$

renders the rescaled Weyl spinor regular at i . Associated to it one has

$$\Theta \mapsto \kappa^{-1} \Theta,$$

with $\kappa = \hat{\kappa}|x|$, $\hat{\kappa}(i) = 1$.

More on the cylinder at spatial infinity



In a suitable gauge:

$$\bar{\mathcal{M}} = \{(\tau, q) \in \mathbb{R} \times \bar{\mathcal{S}} \mid |\tau| \leq 1 + \rho(q)\},$$

$$\mathcal{I} = \{|\tau| < 1, \rho = 0\},$$

$$\mathcal{I}^\pm = \{\tau = \pm 1, \rho = 0\},$$

$$\mathcal{S}^\pm = \{\tau = \pm(1 + \rho(q)), q \in \bar{\mathcal{S}}\}.$$

The conformal propagation equations

Unknowns:

$$v = (e_{AB}^{\mu}, \Gamma_{ABCD}, \Theta_{ABCD}), \quad \phi = (\phi_{ABCD}).$$

The conformal propagation equations

Unknowns:

$$v = (e_{AB}^{\mu}, \Gamma_{ABCD}, \Theta_{ABCD}), \quad \phi = (\phi_{ABCD}).$$

Propagation equations:

$$\partial_{\tau} v = K v + Q(v, v) + L \phi, \quad \text{Complementary equations}$$

$$(\sqrt{2}I + A^0) \partial_{\tau} \phi + A^{\alpha} \partial_{\alpha} \phi = B(\Gamma_{ABCD}) \phi, \quad \text{Bianchi propagation eqns.}$$

The conformal propagation equations

Unknowns:

$$v = (e^{\mu}_{AB}, \Gamma_{ABCD}, \Theta_{ABCD}), \quad \phi = (\phi_{ABCD}).$$

Propagation equations:

$$\partial_{\tau} v = K v + Q(v, v) + L \phi, \quad \text{Complementary equations}$$

$$(\sqrt{2}I + A^0) \partial_{\tau} \phi + A^{\alpha} \partial_{\alpha} \phi = B(\Gamma_{ABCD}) \phi, \quad \text{Bianchi propagation eqns.}$$

Bianchi constraint:

$$F^{\mu} \partial_{\mu} \phi = H(\Gamma_{ABCD}) \phi.$$

Structural properties:

Point of view of the CFE:

- one has a regular setting.

Structural properties:

Point of view of the CFE:

- one has a regular setting.

The cylinder at spatial is a total characteristic:

- The symmetric hyperbolic system of propagation equations reduces to an interior system on \mathcal{I} .
- One can determine $u = (v, \phi)$ on \mathcal{I} from data on \mathcal{I}^0 —no boundary values can be prescribed on \mathcal{I} .

Expansions near the cylinder at spatial infinity:

A general procedure:

- Taking formal derivatives of the equations with respect to the radial coordinate one gets transport equations for $u^{(p)} = \partial_\rho^p u|_{\mathcal{I}}$.

Expansions:

- The functions $u^{(p)}$ on \mathcal{I} can be thought of as the coefficients in the *Taylor-like* expansion:

$$u = \sum_{p \geq 0} \frac{1}{p!} u^{(p)} \rho^p.$$

Note:

- No convergence issues will be discussed here!

The transport equations (I)

Use:

- Provide a way to analyse the fields on \mathcal{I} to any desired degree of precision.
- Allow to relate concepts on null infinity with concepts at spatial infinity.

The transport equations (I)

Use:

- Provide a way to analyse the fields on \mathcal{I} to any desired degree of precision.
- Allow to relate concepts on null infinity with concepts at spatial infinity.

The transport equations in more detail:

$$\begin{aligned}\partial_\tau v^{(p)} &= K v^{(p)} + Q(v^{(0)}, v^{(p)}) + Q(v^{(p)}, v^{(0)}) \\ &\quad + \sum_{j=1}^{p-1} \binom{p}{j} \left(Q(v^{(j)}, v^{(p-j)}) + L^{(j)} \phi^{(p-j)} \right) + L^{(p)} \phi^{(0)}, \\ (\sqrt{2}I + A^{AB} (c_{AB}^0)^{(0)}) \partial_\tau \phi^{(p)} + A^{AB} (c_{AB}^\mu)^{(0)} \partial_\mu \phi^{(p)} &= B(\Gamma_{ABCD}^{(0)}) \phi^{(p)} \\ &\quad + \sum_{j=1}^p \binom{p}{j} \left(B(\Gamma_{ABCD}^{(j)}) \phi^{(p-j)} - A^{AB} (c_{AB}^\mu)^{(j)} \partial_\mu \phi^{(p-j)} \right),\end{aligned}$$

The transport equations (II)

Degeneracy at the critical sets:

- Given

$$(\sqrt{2}I + A^0)\partial_\tau\phi + A^\alpha\partial_\alpha\phi = B(\Gamma_{ABCD})\phi$$

one finds that

$$(\sqrt{2}I + A^0) = \sqrt{2}(1 + \tau, 1, 1, 1, 1 - \tau).$$

- The matrix degenerates on \mathcal{I}^\pm where the *total characteristic* \mathcal{I} meets the *characteristic* \mathcal{J}^\pm transversely.

The transport equations (II)

Degeneracy at the critical sets:

- Given

$$(\sqrt{2}I + A^0)\partial_\tau\phi + A^\alpha\partial_\alpha\phi = B(\Gamma_{ABCD})\phi$$

one finds that

$$(\sqrt{2}I + A^0) = \sqrt{2}(1 + \tau, 1, 1, 1, 1 - \tau).$$

- The matrix degenerates on \mathcal{I}^\pm where the *total characteristic* \mathcal{I} meets the *characteristic* \mathcal{J}^\pm transversely.

Observation:

- The degeneracy of the propagation equations has important consequences.
- In order to gain further insight one can decompose the entries in $u^{(p)}$ in spherical harmonics in order to obtain explicit solutions.

Solving the transport equations (I)

Assumption:

- The lower order solutions $(v^{(i)}, \phi^{(i)})$, $i = 0, \dots, p - 1$ are known.

Solving the transport equations (I)

Assumption:

- The lower order solutions $(v^{(i)}, \phi^{(i)})$, $i = 0, \dots, p-1$ are known.

Decomposition in spherical harmonics:

- If restricted to a suitable class of initial data, then it follows that the entries in $v^{(p)}$ and $\phi^{(p)}$ must have a very definite decomposition in spherical harmonics.
- The latter implies a hierarchy of ordinary differential equations for every spherical harmonic-sector.

Solving the transport equations (I)

Assumption:

- The lower order solutions $(v^{(i)}, \phi^{(i)})$, $i = 0, \dots, p-1$ are known.

Decomposition in spherical harmonics:

- If restricted to a suitable class of initial data, then it follows that the entries in $v^{(p)}$ and $\phi^{(p)}$ must have a very definite decomposition in spherical harmonics.
- The latter implies a hierarchy of ordinary differential equations for every spherical harmonic-sector.

Integration of $v^{(p)}$:

- Unproblematic although computationally involved for increasingly larger p !
- The solutions are polynomial in τ if the lower order solutions $(v^{(i)}, \phi^{(i)})$, $i = 0, \dots, p-1$ are polynomial.

Solving the transport equations (II)

Integration of $\phi^{(p)}$:

- For a given spherical harmonic sector, the solutions can be calculated by solving a 2×2 reduced system:

$$\begin{pmatrix} 1 + \tau & 0 \\ 0 & 1 - \tau \end{pmatrix} \begin{pmatrix} \partial_\tau a_1 \\ \partial_\tau a_3 \end{pmatrix} + A(\tau) \begin{pmatrix} a_1 \\ a_3 \end{pmatrix} = \begin{pmatrix} F_1(\tau) \\ F_3(\tau) \end{pmatrix}.$$

- The matrix A has entries polynomial in τ .
- The functions F_1 and F_3 can be explicitly calculated if $v^{(i)}$, $i = 0, \dots, p$ and $\phi^{(i)}$, $i = 0, \dots, p - 1$ are known.

A first analysis at the critical sets

H. Friedrich, 1998 (time symmetric initial data):

- In the case of time symmetric data, there are sectors for which $F_1 = F_3 = 0$. One can then derive a set of necessary conditions for the spacetime to extend analytically through the critical sets:

$$\mathcal{C}(D_{i_p} \cdots D_{i_1} B_{ij})(i) = 0, \quad p = 0, 1, \dots$$

B_{ij} (Cotton tensor).



A first analysis at the critical sets

H. Friedrich, 1998 (time symmetric initial data):

- In the case of time symmetric data, there are sectors for which $F_1 = F_3 = 0$. One can then derive a set of necessary conditions for the spacetime to extend analytically through the critical sets:

$$\mathcal{C}(D_{i_p} \cdots D_{i_1} B_{ij})(i) = 0, \quad p = 0, 1, \dots$$

B_{ij} (Cotton tensor).



Logarithmic singularities:

- If these conditions are not satisfied, then the data develop a very definite type of logarithmic singularities at the critical sets.

A first analysis at the critical sets

H. Friedrich, 1998 (time symmetric initial data):

- In the case of time symmetric data, there are sectors for which $F_1 = F_3 = 0$. One can then derive a set of necessary conditions for the spacetime to extend analytically through the critical sets:

$$\mathcal{C}(D_{i_p} \cdots D_{i_1} B_{ij})(i) = 0, \quad p = 0, 1, \dots$$

B_{ij} (Cotton tensor).



Logarithmic singularities:

- If these conditions are not satisfied, then the data develop a very definite type of logarithmic singularities at the critical sets.

Question:

- Are these necessary conditions also sufficient?

Conformally flat data (I)

Strategy:

- Specialise further: consider conformally flat initial data sets.

Conformally flat data (I)

Strategy:

- Specialise further: consider conformally flat initial data sets.

Why conformally flat data?

- The regularity conditions are satisfied automatically.
- Conformally flat data is analytically simple.
- Nevertheless, it is a rich class of spacetimes —contains, e.g. data for black hole head on collisions.

Conformally flat data (II)

Time symmetric conformally flat Hamiltonian constraint:

$$\Delta_{\delta}\vartheta = 0.$$

- In a neighbourhood of i the asymptotically Euclidean solutions can be parametrised as

$$\vartheta = \frac{1}{|x|} + W, \quad W(i) = \frac{m}{2}.$$

- Without loss of generality the harmonic function W can be written as

$$W = \frac{m}{2} + \sum_{p=2}^{\infty} \sum_{k=-p}^p w_{p,k} Y_{p,k} |x|^k.$$

- Initial data for the Schwarzschild spacetime corresponds to $W = m/2$.

Computer algebra computations

Computer algebra calculations (JAVK, 2004):

- Further logarithmic singularities at the critical sets have been identified.
- These new class obstructions to the smoothness of null infinity arise from the interaction of the (singular) principal part and the lower order terms in the conformal field equations.



An important observation:

- Particular subsets of the singularities can be eliminated by setting certain pieces of the data to zero.
- A very definite pattern is observed.

Computer algebra computations

Computer algebra calculations (JAVK, 2004):

- Further logarithmic singularities at the critical sets have been identified.
- These new class obstructions to the smoothness of null infinity arise from the interaction of the (singular) principal part and the lower order terms in the conformal field equations.



An important observation:

- Particular subsets of the singularities can be eliminated by setting certain pieces of the data to zero.
- A very definite pattern is observed.

Question:

- Can one prove the pattern observed with the CA experiments?

A theorem

Theorem (JAVK, 2009)

Given a time symmetric initial data set for the Einstein vacuum field equations which is conformally flat, the solution to the regular finite initial value problem at spatial infinity is smooth through the critical sets if and only if the data is exactly Schwarzschildian.

A theorem

Theorem (JAVK, 2009)

Given a time symmetric initial data set for the Einstein vacuum field equations which is conformally flat, the solution to the regular finite initial value problem at spatial infinity is smooth through the critical sets if and only if the data is exactly Schwarzschildian.

Spin offs:

- The particular argument to obtain the proof renders a wealth of information on the algebraic structure of the conformal field equations and their solutions (at \mathcal{I}).
- The proof reveals some further structure of the equations which still needs to be understood.

How to prove the main theorem?

Idea:

- Assume that one has an initial data set which is Schwarzschildian up to order $p = p_\bullet$.

$$W = \frac{m}{2} + \sum_{p=p_\bullet+1}^{\infty} \sum_{k=-p}^p w_{p,k} Y_{p,k} \rho^k.$$

- Integrate the transport equations in as many orders until a singular solution is (hopefully) found.
- Show that this singular solution can be eliminated if and only if

$$w_{p_\bullet+1} = 0.$$

- Conclude with an induction argument.

How to integrate the transport equations?

Observation:

- The procedure to integrate the equations is conceptually straightforward—the equations are linear.
- The main obstacle is one of computational complexity!
 - The computations very quickly go beyond the normal abilities of a human being.
 - Moreover, the computations could very quickly escape the abilities of CA systems.

How to integrate the transport equations?

Observation:

- The procedure to integrate the equations is conceptually straightforward—the equations are linear.
- The main obstacle is one of computational complexity!
 - The computations very quickly go beyond the normal abilities of a human being.
 - Moreover, the computations could very quickly escape the abilities of CA systems.

Crucial point:

- Find a procedure which is completely amenable to computer algebra implementation!!

Computational complexities in solving the equations

Solutions to the Bianchi equations at order $p = p_{\bullet} + 1$

- The solutions for the relevant spherical harmonic sectors ($Y_{p_{\bullet}+1,k}$) are of the form

$$a_{j,p_{\bullet}+1} = C_j w_{p_{\bullet}+1} (1 - \tau)^{p_{\bullet}+3-j} (1 + \tau)^{p_{\bullet}-1+j}.$$

Computational complexities in solving the equations

Solutions to the Bianchi equations at order $p = p_{\bullet} + 1$

- The solutions for the relevant spherical harmonic sectors ($Y_{p_{\bullet}+1,k}$) are of the form

$$a_{j,p_{\bullet}+1} = C_j w_{p_{\bullet}+1} (1 - \tau)^{p_{\bullet}+3-j} (1 + \tau)^{p_{\bullet}-1+j}.$$

Solutions to the complementary equations at order $p = p_{\bullet} + 2$

- Remarkably, it is possible to integrate exactly these equations.
- The result are polynomials in τ which are expressed in terms of hypergeometric functions of various types.

Computational complexities in solving the equations

Solutions to the Bianchi equations at order $p = p_{\bullet} + 1$

- The solutions for the relevant spherical harmonic sectors ($Y_{p_{\bullet}+1,k}$) are of the form

$$a_{j,p_{\bullet}+1} = C_j w_{p_{\bullet}+1} (1 - \tau)^{p_{\bullet}+3-j} (1 + \tau)^{p_{\bullet}-1+j}.$$

Solutions to the complementary equations at order $p = p_{\bullet} + 2$

- Remarkably, it is possible to integrate exactly these equations.
- The result are polynomials in τ which are expressed in terms of hypergeometric functions of various types.

Solutions to the Bianchi equations at order $p = p_{\bullet} + 2?$

- It is possible to prove that the solutions are polynomial in τ , however no formula involving elementary and special functions was found.
- It is therefore not possible to keep explicitly integrating using this approach!!!

Need:

- A representation of the problem rendering solutions which are amenable to solve with straight-forward CA —no use of identities between special functions.

Getting around the computational complexities

Need:

- A representation of the problem rendering solutions which are amenable to solve with straight-forward CA —no use of identities between special functions.

Idea:

- Avoid solving the complementary transport equations!

Getting around the computational complexities

Need:

- A representation of the problem rendering solutions which are amenable to solve with straight-forward CA —no use of identities between special functions.

Idea:

- Avoid solving the complementary transport equations!

How?

- Differentiate the Bianchi transport equations as many times as necessary so that the complementary equations can be substituted in the lower-order source terms...

The case $p = p_\bullet + 2$

Higher order reduced equations:

$$\begin{pmatrix} 1 + \tau & 0 \\ 0 & 1 - \tau \end{pmatrix} \begin{pmatrix} \partial_\tau^4 a_1 \\ \partial_\tau^4 a_3 \end{pmatrix} + A_{p_\bullet+2} \begin{pmatrix} \partial_\tau^3 a_1 \\ \partial_\tau^3 a_3 \end{pmatrix} = \begin{pmatrix} F_{p_\bullet+2}(\tau) \\ F_{p_\bullet+2}(-\tau) \end{pmatrix}$$

with

$$F_{p_\bullet+2}(\tau) = (1 - \tau)^{p_\bullet-3} (1 + \tau)^{p_\bullet-4} Q_{p_\bullet+2}(\tau)$$

and $Q_{p_\bullet+2}$ an explicitly known polynomial of degree 11.

The case $p = p_\bullet + 2$

Higher order reduced equations:

$$\begin{pmatrix} 1 + \tau & 0 \\ 0 & 1 - \tau \end{pmatrix} \begin{pmatrix} \partial_\tau^4 a_1 \\ \partial_\tau^4 a_3 \end{pmatrix} + A_{p_\bullet+2} \begin{pmatrix} \partial_\tau^3 a_1 \\ \partial_\tau^3 a_3 \end{pmatrix} = \begin{pmatrix} F_{p_\bullet+2}(\tau) \\ F_{p_\bullet+2}(-\tau) \end{pmatrix}$$

with

$$F_{p_\bullet+2}(\tau) = (1 - \tau)^{p_\bullet-3}(1 + \tau)^{p_\bullet-4}Q_{p_\bullet+2}(\tau)$$

and $Q_{p_\bullet+2}$ an explicitly known polynomial of degree 11.

Observation:

- One can prove that the polynomial solutions to these equations have to be of the form

$$\partial_\tau^3 a_1 = (1 - \tau)^{p_\bullet-2}(1 + \tau)^{p_\bullet-4}b_{p_\bullet+2}(\tau)$$

$$\partial_\tau^3 a_3 = -(1 + \tau)^{p_\bullet-2}(1 + \tau)^{p_\bullet-4}b_{p_\bullet+2}(\tau),$$

with $b_{p_\bullet+2}$ a polynomial to be calculated.

The case $p = p_\bullet + 2$ (continued)

Observation:

- With this, finding the polynomial solutions to these equations reduces to an exercise of linear algebra.
- The coefficients $\partial_\tau^3 a_1$ and $\partial_\tau^3 a_3$ are found to be polynomial in τ .

The case $p = p_\bullet + 2$ (continued)

Observation:

- With this, finding the polynomial solutions to these equations reduces to an exercise of linear algebra.
- The coefficients $\partial_\tau^3 a_1$ and $\partial_\tau^3 a_3$ are found to be polynomial in τ .

Theorem

If the initial data is Schwarzschildian up to order p_\bullet , then the solutions to the Bianchi transport equations at order $p = p_\bullet + 2$ are polynomial.

The case $p = p_\bullet + 3$ in brief

One proceeds as in the previous case:

$$\begin{pmatrix} 1 + \tau & 0 \\ 0 & 1 - \tau \end{pmatrix} \begin{pmatrix} \partial_\tau^8 a_1 \\ \partial_\tau^8 a_3 \end{pmatrix} + A_{p_\bullet+3} \begin{pmatrix} \partial_\tau^7 a_1 \\ \partial_\tau^7 a_3 \end{pmatrix} = \begin{pmatrix} F_{p_\bullet+3}(\tau) \\ F_{p_\bullet+3}(-\tau) \end{pmatrix}$$

with

$$F_{p_\bullet+3}(\tau) = (1 - \tau)^{p_\bullet-8} (1 + \tau)^{p_\bullet-9} Q_{p_\bullet+3}(\tau)$$

and $Q_{p_\bullet+3}$ an explicitly known polynomial.

The case $p = p_{\bullet} + 3$ in brief

One proceeds as in the previous case:

$$\begin{pmatrix} 1 + \tau & 0 \\ 0 & 1 - \tau \end{pmatrix} \begin{pmatrix} \partial_{\tau}^8 a_1 \\ \partial_{\tau}^8 a_3 \end{pmatrix} + A_{p_{\bullet}+3} \begin{pmatrix} \partial_{\tau}^7 a_1 \\ \partial_{\tau}^7 a_3 \end{pmatrix} = \begin{pmatrix} F_{p_{\bullet}+3}(\tau) \\ F_{p_{\bullet}+3}(-\tau) \end{pmatrix}$$

with

$$F_{p_{\bullet}+3}(\tau) = (1 - \tau)^{p_{\bullet}-8} (1 + \tau)^{p_{\bullet}-9} Q_{p_{\bullet}+3}(\tau)$$

and $Q_{p_{\bullet}+3}$ an explicitly known polynomial.

Theorem

If the initial data is Schwarzschildian up to order p_{\bullet} , then the solutions to the Bianchi transport equations at order $p = p_{\bullet} + 3$ are polynomial.

The interesting case: $p = p_{\bullet} + 4$

A lengthy Maple calculation renders:

$$\begin{pmatrix} 1 + \tau & 0 \\ 0 & 1 - \tau \end{pmatrix} \begin{pmatrix} \partial_{\tau}^{12} a_1 \\ \partial_{\tau}^{12} a_3 \end{pmatrix} + A_{p_{\bullet}+4} \begin{pmatrix} \partial_{\tau}^{11} a_1 \\ \partial_{\tau}^{11} a_3 \end{pmatrix} = \begin{pmatrix} F_{p_{\bullet}+4}(\tau) \\ F_{p_{\bullet}+4}(-\tau) \end{pmatrix}$$

with

$$F_{p_{\bullet}+4}(\tau) = (1 - \tau)^{p_{\bullet}-13} (1 + \tau)^{p_{\bullet}-14} Q_{p_{\bullet}+4}(\tau)$$

and $Q_{p_{\bullet}+4}$ an explicitly known polynomial.

The interesting case: $p = p_\bullet + 4$

A lengthy Maple calculation renders:

$$\begin{pmatrix} 1 + \tau & 0 \\ 0 & 1 - \tau \end{pmatrix} \begin{pmatrix} \partial_\tau^{12} a_1 \\ \partial_\tau^{12} a_3 \end{pmatrix} + A_{p_\bullet+4} \begin{pmatrix} \partial_\tau^{11} a_1 \\ \partial_\tau^{11} a_3 \end{pmatrix} = \begin{pmatrix} F_{p_\bullet+4}(\tau) \\ F_{p_\bullet+4}(-\tau) \end{pmatrix}$$

with

$$F_{p_\bullet+4}(\tau) = (1 - \tau)^{p_\bullet-13} (1 + \tau)^{p_\bullet-14} Q_{p_\bullet+4}(\tau)$$

and $Q_{p_\bullet+4}$ an explicitly known polynomial.

Ansatz:

One can prove that all polynomial solutions to the reduced system have to be of the form:

$$\begin{aligned} \partial_\tau^{11} a_1 &= (1 - \tau)^{p_\bullet-12} (1 + \tau)^{p_\bullet-14} b_{p_\bullet+4}(\tau) \\ \partial_\tau^{11} a_3 &= -(1 + \tau)^{p_\bullet-12} (1 - \tau)^{p_\bullet-14} b_{p_\bullet+4}(\tau), \end{aligned}$$

with $b_{p_\bullet+4}$ a polynomial of degree 29.

There are no polynomial solutions!

Observations:

- A lengthy but straight-forward CA calculation shows there are no polynomial solutions.
- A further general argument shows that the solution is generated by polynomials in τ and $\ln(1 \pm \tau)$.

There are no polynomial solutions!

Observations:

- A lengthy but straight-forward CA calculation shows there are no polynomial solutions.
- A further general argument shows that the solution is generated by polynomials in τ and $\ln(1 \pm \tau)$.

Theorem

If the initial data is Schwarzschildian up to order $p = p_\bullet$, then the solutions to the Bianchi transport equations at order $p_\bullet + 4$ develop logarithmic singularities at $\tau = \pm 1$ unless

$$w_{p_\bullet+1,k} = 0 \quad \text{or} \quad m = 0.$$

The singular solutions are of class

$$C^\omega(-1, 1) \cap C^{p_\bullet+3}[-1, 1].$$

Concluding the proof:

An inductive argument allows to close the proof of the main result.

Back to the main theorem...

Concluding the proof:

An inductive argument allows to close the proof of the main result.

Theorem

Given a time symmetric initial data set for the Einstein vacuum field equations which is conformally flat, the solution to the regular finite initial value problem at spatial infinity is smooth through the critical sets if and only if the data is exactly Schwarzschildian.

Remarks:

Information about structural properties:

- The analysis leading to the main theorem provides very detailed information about the behaviour of the solutions to the conformal field equations at spatial infinity.

Information about structural properties:

- The analysis leading to the main theorem provides very detailed information about the behaviour of the solutions to the conformal field equations at spatial infinity.

Structural properties of the CFE:

- The logarithmic singularities in the solutions appear at higher orders than what is to be expected from general arguments.
 - Which structural properties of the equations are responsible for this?
Connection with *differential Galois theory*?
- Paradoxically, the extra regularity makes the analysis much more computationally challenging.

Non-smoothness of null infinity?

Question:

do the logarithmic singularities at the critical sets propagate along null infinity?

Non-smoothness of null infinity?

Question:

do the logarithmic singularities at the critical sets propagate along null infinity?

Expectation:

possibly yes, however this still needs to be proved!

Bringing the main result into broader context:

- The Schwarzschild spacetime is the only static spacetime admitting conformally flat slices.

Bringing the main result into broader context:

- The Schwarzschild spacetime is the only static spacetime admitting conformally flat slices.

A possible generalisation:

- Smoothness at the critical sets implies staticity in a neighbourhood of spatial infinity (currently being analysed) —evidence found in (JAVK, 2004).
 - The challenge is to find a parametrisation of time symmetric initial data sets from where it is simple to identify static data.