Regularity conditions at spatial infinity revisited

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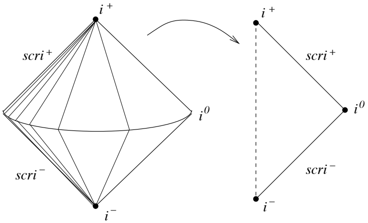
Outline

Asymptotically simple spacetimes



Definition

A smooth (C^∞) spacetime $(\widetilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ is called asymptotically simple if there is another manifold $(\mathcal{M}, g_{\mu\nu})$ such that:

(i) $\widetilde{\mathcal{M}}$ is an open submanifold of $\mathcal{M}$ with smooth boundary $\partial\widetilde{\mathcal{M}} = \mathscr{I}$;

(ii) there is a Ω on $\mathcal{M}$ such that $g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$ on $\widetilde{\mathcal{M}}$ and so that $\Omega = 0$, $d\Omega \neq 0$ on $\mathscr{I}$;

(iii) every null geodesic in $\widetilde{\mathcal{M}}$ acquires a future and a past endpoint on $\mathscr{I}$;

(iv) $\widetilde{R}_{\mu\nu} = 0$.

How to construct asymptotically simple spacetimes?

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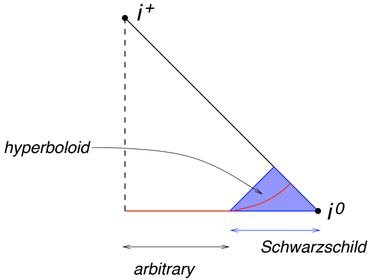


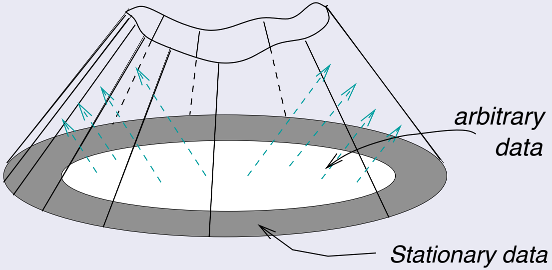
- A large family of spacetimes that are asymptotically simple has been shown to exist (Chruściel & Delay):
  - initial data which is asymptotically Schwarzschildean near infinity —constructed using the Corvino-Schoen-Chruściel-Delay gluing construction;
  - semiglobal existence results for small (close to Minkowski) hyperboloidal data.

Rigidity conjecture

There is some evidence that asymptotically simple spacetimes are rigid in a certain sense:

Conjecture

If the development of asymptotically Euclidean initial data admits both a smooth future null infinity and a smooth past null infinity, then the initial data is stationary in a neighbourhood of infinity.

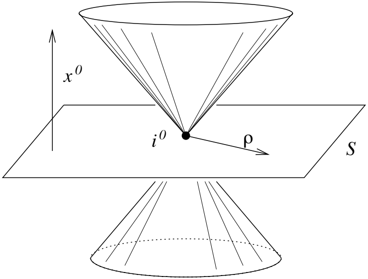


Objective:

Want to gain more insight into this conjecture...

Outline

Spacetime near spatial infinity

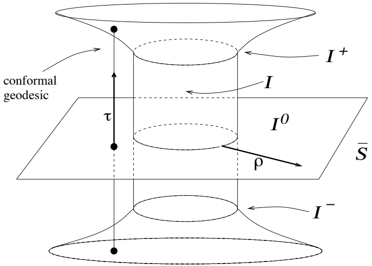


Strategy

- Analyse with care the structure of spacetime close to spatial infinity.
- The spacetime is obtained as the development of suitable initial data:
  - accordingly, assumptions on the initial data $(\tilde{h}_{ij}, \tilde{\chi}_{ij})$ are required,

$$\tilde{h}_{ij} \sim \left(1 + \frac{2m}{\tilde{r}}\right)\delta_{ij} + \sum_{k\geq 2} \frac{\tilde{h}^{(k)}_{ij}}{\tilde{r}^k}, \qquad \tilde{\chi}_{ij} \sim \sum_{k\geq 2} \frac{\tilde{\chi}^{(k)}_{ij}}{\tilde{r}^k}$$

The cylinder at spatial infinity



This representation of spatial infinity allows to formulate a:

regular finite initial value problem at spatial infinity

- The equations, field variables and data are regular.
- The location of the conformal boundary is known *a priori* and can be read from the initial data.

On the construction of the cylinder at spatial infinity

Blow up of i:

- achieved by considering a submanifold of $SU(\mathcal{S})$, the bundle of normalised spin frames over $\mathcal{S}$ with structure group $SU(2)$.

Conformal geodesics:

- the use of conformal Gaussian coordinates provides a canonical conformal factor, Θ, which can be calculated from the solutions to the constraint equations. The location of the conformal boundary is known *a priori*!

Rescaling of the spinor dyad:

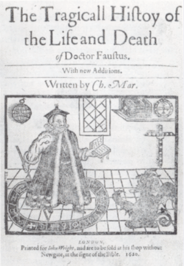
$$\delta_A \mapsto \kappa^{1/2}\delta_A,$$

allows to render (*inter alia*) the rescaled Weyl spinor regular at i —more on that later! Associated to it one has

$$\Theta \mapsto \kappa^{-1}\Theta,$$

with $\kappa = \hat{\kappa}|x|$, $\hat{\kappa}(i) = 1$.

Spinors



Space-spinors:

- The calculations described here are best performed using a space-spinor formalism —$SU(2)$-spinors: algebraic advantages!

A small dictionary:

$$X_i \longrightarrow X_{AB} = X_{(AB)},$$

$$\text{trace-free part}(X_{i_p \cdots i_1}) \longrightarrow X_{(A_p B_p \cdots A_1 B_1)}$$

$$\epsilon_i{}^{jk} X_j Y_k \longrightarrow -\mathrm{i}\sqrt{2} X^E{}_A Y_{BE}.$$

The conformal field equations

Unknowns:

- The components of the frame, connection, and Ricci tensor

$$v = (c^\mu_{AB}, \Gamma_{ABCD}, \Phi_{ABCD}).$$

- The components of the rescaled Weyl spinor

$$\breve{\phi} = (\breve{\phi}_0, \breve{\phi}_1, \breve{\phi}_2, \breve{\phi}_3, \breve{\phi}_4).$$

Evolution equations:

$$\partial_\tau v = Kv + Q(v,v) + L\breve{\phi},$$
$$A^0 \partial_\tau \breve{\phi} + A^\alpha \partial_\alpha \breve{\phi} = B(\Gamma_{ABCD})\breve{\phi},$$

Initial data for the conformal field equations I

Conformal Ansatz

Construct a solution $(\tilde{h}_{ij}, \tilde{\chi}_{ij})$ of the physical constraint equations assuming $\tilde{\mathcal{S}}$, $\mathcal{S}$ maximal and setting:

$$
\begin{aligned}
&\tilde{\chi} = \chi = 0, \\
&\tilde{h}_{ij} = \vartheta^4 h_{ij}, \quad \tilde{\chi}_{ij} = \vartheta^2 \chi_{ij} = \vartheta^{-2}\psi_{ij}, \\
&\psi_{ij} = \mathring{\psi}_{ij} + D_i v_j + D_j v_i - \frac{2}{3} h_{ij} D^k v_k.
\end{aligned}
$$

The equations for ϑ and v_i

The equations to be solved are

$$
\left(D_k D^k - \frac{1}{8} r \right) \vartheta = \frac{1}{8} \psi_{ij}\psi^{ij}\vartheta^{-7},
$$

$$
D_k D^k v_j + \frac{1}{3} D_j D_i v^i + r_{ji} v^i = -D^i \mathring{\psi}_{ij}.
$$

Initial data for the conformal field equation II

Once (ϑ, ψ_{ij}) are known, use the *conformal constraint equations* to calculate the initial data for the propagation equations.

For example:

$$\breve{\phi}_{ABCD} = \kappa^3 \left(w_{ABCD} + \mathrm{i} w^*_{ABCD} \right)$$

with

$$w_{ABCD} = \Omega^{-2} D_{(AB} D_{CD)} \Omega + \Omega^{-1} \left(s_{ABCD} - \chi^{EF}{}_{(AB} \chi_{CD)EF} \right),$$

$$w^*_{ABCD} = -\mathrm{i}\sqrt{2}\Omega^{-1} D^F{}_{(A} \chi_{BCD)F},$$

where

$$\Omega = \vartheta^{-2}.$$

Transport equations

Structural property of propagation equations:

- The conformal propagation equations reduce, upon evaluation at $\mathcal{I}$ —i.e. $\rho = 0$—, to a system of interior equations: transport equations.
- The transport equations allow to calculate

$$\phi_j^{(p)} = \partial_\rho^p \phi_j |_{\rho=0}.$$

- The transport equations are linear and form a hierarchy: knowledge of $\phi_j^{(p')}$ (and the other field unknowns!) for $0 \leq p' \leq p-1$ allows to calculate $\phi_j^{(p)}$ —modulo calculational complexities.

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Strategy:

- Use the transport equation to analyse the behaviour of the solutions to the conformal field equations at the critical sets $\mathcal{I}^{\pm}$:
  - essentially one is constructing a formal asymptotic expansions which allows to relate properties of the initial with behaviour at $\mathcal{I}^{\pm}$.

Decomposition in "spherical harmonics"

Under the appropriate assumptions on the initial data one can write

$$\phi_j^{(p)} = \sum_{q=|2-q|}^{p} \sum_{k=0}^{2q} a_{j,p;2q,k} T_{2q}{}^{k}{}_{q-2+j}, \qquad \phi_j^{(p)} = \partial_\rho^p \phi_j|_{\rho=0}$$

with $a_{j,p;2q,k} = a_{j,p;2q,k}(\tau)$.

The functions $T_k{}^l{}_m$ form an orthonormal basis in $L^2(SU(2))$:

$$(-\mathrm{i})^{s+2n-m}\sqrt{\frac{2n+1}{4\pi}}T_{2n}{}^{n-m}{}_{n-s} \longrightarrow {}_sY_{nm}.$$

The transport equations imply a hierarchy of ODE's for each mode:

- unfolding of the evolution process, which can be analysed in all detail and to any desired order —in principle!

Logarithmic divergences at the critical sets

Here only the transport equations for the highest possible sectors in $\phi_j^{(p)}$ will be considered:

$$a_{j,p;2p,k}(\tau)T_{2p}{}^{k}{}_{q-2+j}, \quad j=0,\ldots,4 \quad k=0,\ldots,2p$$

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The solutions:

$$a_{0,p;2p,k}(\tau) = (1-\tau)^{p+2}(1+\tau)^{p-2}\left(C_0 + C_1 \int_0^\tau \frac{\mathrm{d}s}{(1+s)^{p-1}(1-s)^{p+3}}\right),$$

$$a_{4,p;2p,k}(\tau) = (1+\tau)^{p+2}(1-\tau)^{p-2}\left(C_0 + C_1 \int_0^\tau \frac{\mathrm{d}s}{(1-s)^{p-1}(1+s)^{p+3}}\right).$$

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The solutions are regular at $\tau=\pm1$ if and only if:

$$a_{0,p;2p,k}(0) = a_{4,p;2p,k}(0), \quad k=0,\ldots,2p.$$

Regularity conditions

The regularity condition

$$a_{0,p;2p,k}(0) = a_{4,p;2p,k}(0), \quad k = 0, \ldots, 2p,$$

is, in principle, a condition on the physical initial data

$$\tilde{h}_{ij} = \Omega^{-2} h_{ij}, \quad \tilde{\chi}_{ij} = \Omega \left(D_i v_j + D_j v_i - \frac{2}{3} D^k v_k h_{ij} + \mathring{\psi}_{ij} \right)$$

as

$$\breve{\phi}_{ABCD} = \kappa^3 \Omega^{-2} D_{(AB} D_{CD)} \Omega + \kappa^3 \Omega^{-1} \left(s_{ABCD} - \chi^{EF}{}_{(AB} \chi_{CD)EF} \right) + \kappa^3 \mathrm{i} \sqrt{2} \Omega^{-1} D^F{}_{(A} \chi_{BCD)F}.$$

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$$+ \kappa^3 \mathrm{i}\sqrt{2} \Omega^{-1} D^F{}_{(A} \chi_{BCD)F}.$$

Question:

- Can one write it purely in terms of the freely specifiable data?:

$$h_{ij}, \quad \mathring{\psi}_{ij}$$

Outline

Time symmetric data

Assume:

- the data is time symmetric, $\chi_{ij} = 0$.
- the conformal metric h_{ij} is analytic at i

- Under these assumptions, H. Friedrich (1998) has shown how the regularity condition

$$a_{0,p;2p,k}(0) = a_{4,p;2p,k}(0), \quad k = 0, \ldots, 2p, \quad p = 2, 3, \ldots$$

can be rewritten as a condition on the freely specifiable data. Namely, one has that

$$D_{(A_{p'}B_{p'}} \cdots D_{A_1B_1} b_{ABCD)}(i) = 0, \quad p' = 0, 1, \ldots.$$

Coton tensor:

$$b_{ABCD} = D^H{}_{(A}s_{BCD)H}, \quad \text{"curl" of tracefree part of the Ricci tensor}$$

About the regularity condition in terms of the Bach tensor

Tensorial version:

the condition

$$D_{(A_{p'}B_{p'}} \cdots D_{A_1B_1} b_{ABCD)}(i) = 0, \quad p' = 0, 1, \ldots,$$

can be rewritten tensorially as

$$\mathcal{C}(D_{i_p} \cdots D_{i_1} b_{ij})(i) = 0, \quad p = 0, 1, \ldots$$

with $\mathcal{C}$ denoting the symmetric-tracefree part thereof.

Analyticity:

- Analyticity is, in principle, non-essential. An assumption to simplify the calculations. One could do C^k or C^∞.
- Analyticity allows performing calculations on the complex null cone through i. Otherwise have to carry lengthy inductive arguments!

Outline

Assumptions on the initial data I

Conformal metric:

As before assume that the conformal metric, h_{ij}, is analytic

Freely specifiable second fundamental form:

$$\mathring{\psi}_{ij} = \frac{A}{r^3}\left(3\frac{x_i x_j}{r^2} - \delta_{ij}\right) + \frac{3}{r^3}\left(\frac{x_i x_l J_m \epsilon_j{}^{ml}}{r^2} + \frac{x_j x_l J_m \epsilon_i{}^{ml}}{r^2}\right)$$
$$+\frac{3}{2r^3}\left(Q_i x_j + Q_j x_i - \left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) Q^k x_k\right) + \mathring{\psi}_{ij}[\lambda]$$
$$-(\text{trace terms})$$

- The tensor $\mathring{\psi}_{ij}[\lambda] = \mathcal{O}(1/r^2)$ is determined entirely by a complex function λ of the form

$$\lambda = \lambda^{(1)} + \frac{1}{r}\lambda^{(2)},$$

with $\lambda^{(1)}$ and $\lambda^{(1)}$ having real and imaginary parts which are C^ω.

Assumptions on the initial data II

Real and imaginary parts of λ:

- Write

$$\lambda = \lambda^{(R)} + \mathrm{i}\lambda^{(I)}$$

with $\lambda^{(R)} = \overline{\lambda^{(R)}}$ and $\lambda^{(I)} = -\overline{\lambda^{(I)}}$.

Interpretation:

$$\mathring{\psi}_{ij}[\lambda^{(R)}] \longrightarrow \text{(higher mass multipoles)}$$
$$\mathring{\psi}_{ij}[\lambda^{(I)}] \longrightarrow \text{(higher angular momentum multipoles)}$$

Ancillary results

Dain & Friedrich (2001):

For the class of freely specifiable data under consideration the conformal factor ϑ solving the Licnerowicz equation admits an expansion in terms of powers of r in a neighbourhood $\mathcal{B}_a(i)$ of i.

Massless and massive parts of $\Omega = \vartheta^{-2}$:

- In $\mathcal{B}_a(i)$ one has the following parametrisation of $\Omega = \vartheta^{-2}$:

$$\Omega = \frac{|x|^2}{(U + |x|W)^2}.$$

- The part of Ω determined solely by setting U renders the *massless* part Ω'.
  - Ω' is analytic if h_{ij} is analytic.
- $\Omega^\bullet = \Omega - \Omega'$ renders massive part of Ω.
  - $W(i) = m/2$, with m the ADM mass.

Decomposition of ϕ_{ABCD} I

- The spinor ϕ_{ABCD} can be constructed from Ω, s_{ABCD} and $\chi_{ABCD} = \Omega^2\psi_{ABCD}$ using the formula

$$\phi_{ABCD} = w_{ABCD} + \mathrm{i}w^*_{ABCD},$$

with

$$w_{ABCD} = \Omega^{-2}D_{(AB}D_{CD)}\Omega + \Omega^{-1}s_{ABCD} + \Omega^{-1}\left(\chi^{EF}{}_{EF}\chi_{(ABCD)} - \chi^{EF}{}_{(AB}\chi_{CD)EF}\right),$$

$$w^*_{ABCD} = -\mathrm{i}\Omega^{-1}\sqrt{2}D^F{}_{(A}\chi_{BCD)F},$$

w_{ABCD} the electric part and w^*_{ABCD} the magnetic part.

- The parametrisation $\Omega = \Omega^\bullet + \Omega'$ allows a decomposition of ϕ_{ABCD} into massless (ϕ'_{ABCD}) and massive ($\phi^\bullet_{ABCD}$) parts:

$$\phi_{ABCD} = \phi'_{ABCD} + \phi^\bullet_{ABCD}.$$

Decomposition of ϕ_{ABCD} II

Refining the decomposition

- It turns out convenient further to write

$$w^{*\prime}_{ABCD} = w^{*\prime}_{ABCD}[A, J, Q, \lambda^{(I)}] + w^{*\prime}_{ABCD}[\lambda^{(R)}].$$

About the massless part of ϕ_{ABCD}

- A calculation renders

$$\phi'_{ABCD} = \frac{1}{|x|^2}\Big(U^2 D_{(AB}D_{CD)}|x|^2 - 4U D_{(AB}|x|^2 D_{CD)}U - 2|x|^2 D_{(AB}D_{CD)}U$$

$$+6|x|^2 D_{(AB}U D_{CD)}U + |x|^2 U^2 s_{ABCD}\Big)$$

$$-\frac{|x|^6}{U^6}\psi^{EF}{}_{(AB}\psi_{CD)EF} + \frac{2\sqrt{2}}{U^2}D^F{}_{(A}|x|^2\psi_{BCD)F}$$

$$-4\sqrt{2}D^F{}_{(A}U\psi_{BCD)F} + \frac{\sqrt{2}|x|^2}{U^2}D^F{}_{(A}\psi_{BCD)F}.$$

- Under suitable assumptions on h_{ij}, ϕ'_{ABCD} can be extended to an analytic field on $\mathcal{B}_a(i)$ —more on that later!

About the massive part of ϕ_{ABCD}

- The expression for the massive part $\phi^{\bullet}_{ABCD}$ is much lengthier and not very illuminating. Will not be reproduced here!
- If $m \neq 0$ then

$$\phi^{\bullet}_{ABCD} = \mathcal{O}\left(\frac{1}{|x|^3}\right)$$

as $|x| \to 0$.

Analysis of the expansions of ϕ'_{ABCD} and $\phi^{\bullet}_{ABCD}$

Regularity condition:

$$a_{0,p;2p,k}(0) = a_{4,p;2p,k}(0), \quad p = 2, 3, \ldots \quad k = 0, \ldots, 2p$$

Write in terms of free specifiable data:

- Perform a detailed analysis of the expansions of $\phi^{\bullet}_{ABCD}$ and ϕ'_{ABCD}:
  - An analysis of symmetry/antisymmetry properties in the spherical harmonic expansions reveals that only

    $$w'_{ABCD} = 0, \quad w^{*\prime}_{ABCD}[\lambda^{(R)}]$$

    turn out to be relevant for the regularity condition.
- Using general results on expansions in terms of spherical harmonics rewrite the conditions as spinorial/tensorial expressions.

An auxiliary spinor/tensor

c_{ij} and c_{ABCD}

- One has $\chi_{ij} = \Omega^2\psi_{ij}$. Define c_{ij} via

$$
\begin{aligned}
c_{ij} &= D_k \chi_{l(i}\epsilon^{kl}{}_{j)}, \\
&= -\Omega w^*_{ij}.
\end{aligned}
$$

- In spinorial notation

$$c_{ABCD} = -\mathrm{i}\sqrt{2}D^E{}_{(A}\chi_{BCD)E}.$$

- The spinors

$$c_{ABCD}[\lambda^{(R)}], \quad c_{ABCD}[\lambda^{(I)}],$$

etc. have the obvious meaning.

The main result I

The main result is the following:

Theorem

For the class of data under consideration, the solution to the regular finite initial value problem at spatial infinity is smooth through $\mathcal{I}^\pm$ *only if*

$$D_{(A_pB_p}\cdots D_{A_1B_1}b_{ABCD)}(i) = 0, \quad p = 0, 1, 2, \ldots$$
$$D_{(A_qB_q}\cdots D_{A_1B_1}c_{ABC|E|)}{}^{E}[\lambda^{(R)}](i) = 0, \quad q = 0, 1, 2, \ldots$$

are satisfied by the data. If any of the above conditions are violated at some order p *or* q*, then the solution will develop logarithmic singularities at* $\mathcal{I}^\pm$*.*

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For the class of data under consideration, the solution to the regular finite initial value problem at spatial infinity is smooth through $\mathscr{I}^{\pm}$ *only if*

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are satisfied by the data. If any of the above conditions are violated at some order p *or* q*, then the solution will develop logarithmic singularities at* $\mathscr{I}^{\pm}$.

Conditions on free data?

- It can be verified that the above are conditions only on the free initial data. For the condition on b_{ABCD} it is more or less direct. For the one on c_{ABCD} it requires more work.

The main result II

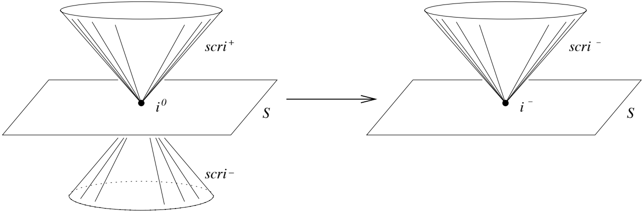
Tensorial transcription of the regularity conditions:

$$\mathcal{C}(D_{i_p} \cdots D_{i_1} b_{ij})(i) = 0, \quad p = 0, 1, 2, \ldots$$
$$\mathcal{C}(D_{i_q} \cdots D_{i_1} c_{ij}[\lambda^{(R)}]\epsilon^{i_1 i}{}_k)(i) = 0, \quad q = 0, 1, 2, \ldots$$

where $\mathcal{C}$ denotes the symmetric, trace-free part of the relevant tensor.

Outline

Another perspective I



Consider a point i and a scalar field Ω on $\mathcal{S}$ such that:

$$\Omega(i) = 0, \quad D_i\Omega(i) = 0, \quad D_iD_j\Omega(i) = 2h_{ij}.$$

- The point i can be regarded as the spatial infinity (i^0) of the development of data on $\mathcal{S}$.
- Alternatively, one can regard i as the past timelike infinity (i^-) of a spacetime.

Another perspective II

Massless Cauchy data

$$\Omega' = \frac{|x|^2}{U^2} \quad \text{—cfr.} \quad \Omega = \frac{|x|^2}{(U + |x|W)^2}.$$

- Ω' is analytic if h_{ij} is analytic.
- Massless Cauchy initial data in a neighbourhood of i can be used to construct purely radiative data on past null infinity (near i^-) for the conformal field equations.

Question:

- What extra conditions are required on h_{ij}, ψ_{ij} so that the initial data for the conformal field equations are analytic near i? (and hence also near i^0 and i^-)

Another perspective III

Regularity conditions for "purely radiative" data:

- The relevant analysis renders the following conditions

$$D_{(A_pB_p}\cdots D_{A_1B_1}\tilde{b}_{ABCD)}(i)=0, \quad p=0,1,2,\ldots$$
$$D_{(A_qB_q}\cdots D_{A_1B_1)}c_{ABCD}(i)=0, \quad q=0,1,2,\ldots$$

  where c_{ABCD} as before and

$$\tilde{b}_{ABCD}=b_{ABCD}+(\text{extra terms quadratic in }\chi_{ABCD}).$$

- If ψ_{ij} (and hence χ_{ij}) is constructed as described here then the condition on $\tilde{b}_{ABCD}$ is equivalent to

$$D_{(A_pB_p}\cdots D_{A_1B_1}b_{ABCD)}(i)=0, \quad p=0,1,2,\ldots$$

Another perspective IV

But...

- S. Dain has shown that Kerr initial data does not satisfy the condition on c_{ABCD}!
- Note that

$$D_{(A_qB_q}\cdots D_{A_1B_1)}c_{ABCD}(i)=0 \Rightarrow D_{(A_qB_q}\cdots D_{A_1B_1}c_{ABC|E|)}{}^{E}(i)=0,$$

but

$$D_{(A_qB_q}\cdots D_{A_1B_1}c_{ABC|E|)}{}^{E}(i)=0 \not\Rightarrow D_{(A_qB_q}\cdots D_{A_1B_1)}c_{ABCD}(i)=0.$$

—cfr. decomposition in irreducible symmetric spinors.

Outline

Stationary data?

Strictly stationary data (i.e. non-static) is not included in our class of initial data:

- Stationary conformal metrics are non-analytic at i.

$$h_{ij} = h_{ij}^{(1)} + r^3 h_{ij}^{(2)}, \quad h_{ij}^{(1)},\ h_{ij}^{(2)} \ \text{analytic}$$

- h_{ij} is $C^{2,\alpha}$.

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Strictly stationary data (i.e. non-static) is not included in our class of initial data:

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- h_{ij} is $C^{2,\alpha}$.

To do:

- Generalise the analysis given here to such conformal metrics.
- In addition, one needs

$$\psi_{ij} = r^{-5}\psi^{(1)}_{ij} + r^{-4}\psi^{(2)}_{ij}, \quad \psi^{(1)}_{ij} = \mathcal{O}(r^2),\ \psi^{(2)}_{ij} = \mathcal{O}(r^2) \ \text{analytic.}$$

Conjectures?

Insights:

- Preliminary calculations seem to indicate that a similar result holds for the more general class of initial data —the calculations are lengthier!
- The leading terms of b_{ABCD} and c_{ABCD} satisfy the condition.

To prove:

- Stationary data satisfy the regularity conditions for b_{ABCD} and c_{ABCD} at all orders.