A conjecture on the behaviour of the development of time symmetric, conformally flat initial data at spatial infinity

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Outline

Asymptotically simple spacetimes

Definition

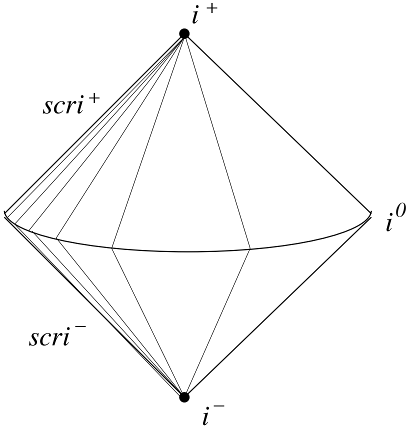
A smooth (C^∞) spacetime $(\widetilde{\mathcal{M}}, \widetilde{g}_{\mu\nu})$ is called asymptotically simple if there is another manifold $(\mathcal{M}, g_{\mu\nu})$ such that:

(i) $\widetilde{\mathcal{M}}$ is an open submanifold of $\mathcal{M}$ with smooth boundary $\partial\widetilde{\mathcal{M}} = \mathscr{I}$;

(ii) there is a Ω on $\mathcal{M}$ such that $g_{\mu\nu} = \Omega^2 \widetilde{g}_{\mu\nu}$ on $\widetilde{\mathcal{M}}$ and so that $\Omega = 0$, $d\Omega \neq 0$ on $\mathscr{I}$;

(iii) every null geodesic in $\widetilde{\mathcal{M}}$ acquires a future and a past endpoint on $\mathscr{I}$;

(iv) $\widetilde{R}_{\mu\nu} = 0$.



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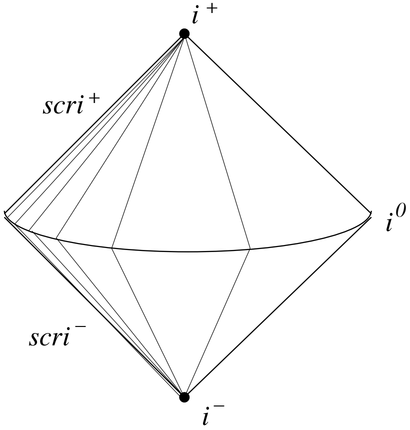
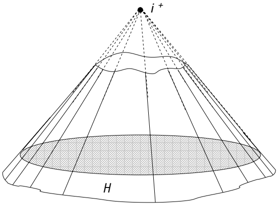


Question:

How to construct, starting from Cauchy data, asymptotically simple spacetimes?

The hyperboloidal initial value problem



Theorem (Friedrich, 1986)

Given the hyperboloidal initial value problem, for suitably smooth initial data close to Minkowski hyperboloidal data one has semi-global existence. The development has the same asymptotic structure as Minkowski spacetime.

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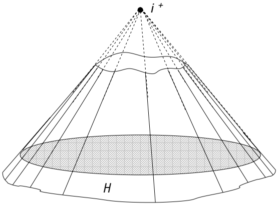

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Remarks:

- The location of the intersection of the hyperboloid with null infinity is irrelevant for the theorem.
- However, the hyperboloid can not be pushed down to spatial infinity, i^0.
- The smoothness of null infinity seems to depend on what happens in an arbitrary small neighbourhood of spatial infinity.

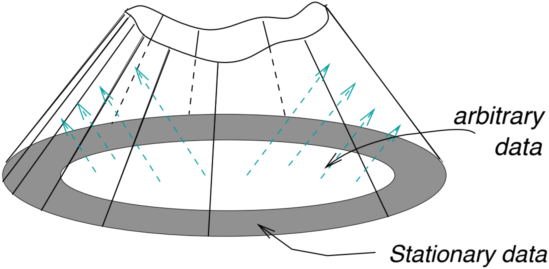
Leitmotiv:

Understand the role of spatial infinity for the smoothness of null infinity.

A vague conjecture...

Evidence:

spacetimes with Cauchy slices which are stationary in a neighbourhood of spatial infinity play special role among asymptotically simple spacetimes.



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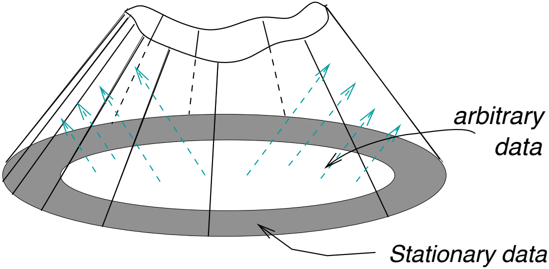
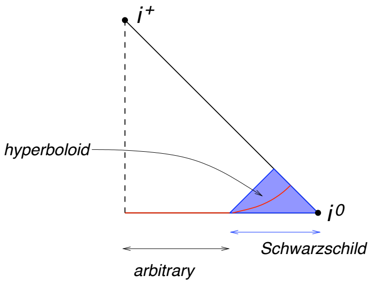


Objective of this talk:

- The purpose of this talk is to discuss how this vague idea can be formulated in a more precise way!

Spacetimes which are stationary in a neighbourhood of i^0

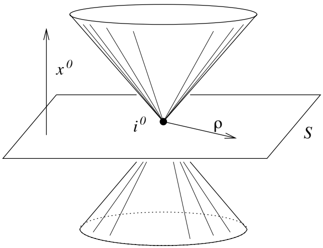


Ingredients to construct an asymptotically simple spacetime:

- Construct Cauchy data using the Corvino-Schoen-Chruściel-Delay gluing techniques.
- Semi-global existence of the development and the smoothness of null infinity follows from Friedrich's existence result for hyperboloidal data.

Outline

The standard representation of spatial infinity



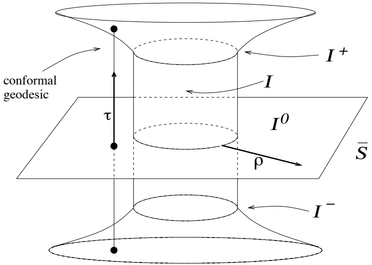
Spatial infinity as a point:

Discuss asymptotics in a conformally rescaled spacetime $(\mathcal{M}, g_{\mu\nu})$,

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$$

- This representation is not appropriate since relevant objects like the Weyl tensor are in general singular at i^0.
- Furthermore, one would like to analyse questions regarding directionally dependent limits.

The cylinder at spatial infinity



An alternative representation (H.Friedrich).

- Blow up the point i^0.
- Use conformal Gaussian coordinates. One has that:

$$\Theta = \frac{1}{\kappa\vartheta^2}\left(1 - \tau^2\kappa^2\left(\frac{D_i\vartheta D^i\vartheta}{\vartheta^2}\right)\right), \quad \text{(Conformal factor)}$$

$$I^\pm = \{\tau = \pm 1, \rho = 0\} \quad \text{(Critical sets)}$$

- Rescale conveniently the frame to regularise singular spinors.

The regular initial value problem at spatial infinity

One more ingredient:

together with the conformal Einstein field equations, this representation allows to formulate the

regular finite initial value problem near spatial infinity.

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regular finite initial value problem near spatial infinity.

Advantages:

- The location of the conformal boundary (infinity) is known *a priori*.
- The equations and data are regular up to the conformal boundary.
- One can obtain statements about the smoothness of solutions near the sets $\mathscr{I}^{\pm} \cup I^{\pm}$.
- Provides a way of unfolding the time evolution near I so that it can be analysed to the required detail.

The unknowns:

Work with spinors (not essential but convenient!).

- The components of the frame, connection, and Ricci tensor

$$v = (c^{\mu}_{AB}, \Gamma_{ABCD}, \Phi_{ABCD}).$$

- The components of the Weyl spinor

$$\phi = (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4).$$

The equations:

- The evolution equations are given by:

$$\partial_\tau v = Kv + Q(v,v) + L\phi,$$
$$A^0\partial_\tau \phi + A^\alpha \partial_\alpha \phi = B(\Gamma_{ABCD})\phi. \quad \text{(Bianchi subsystem)}$$

- There are in addition 3 Bianchi constraint equations.

Initial data for the conformal propagation equations

The Hamiltonian and momentum constraints:

Let be given an initial data set for the vacuum Einstein equations $(\tilde{\mathcal{S}}, \tilde{h}_{ij}, \tilde{\chi}_{ij})$ satisfying

$$\tilde{r} - \tilde{\chi}^2 + \tilde{\chi}_{ij}\tilde{\chi}^{ij} = 0,$$
$$\tilde{D}^i\tilde{\chi}_{ij} - \tilde{D}_j\tilde{\chi} = 0,$$

- Initial data for the conformal propagation equations can be calculated using the

conformal constraint equations

Remark:

- The Hamiltonian and momentum constraints will be solved under the assumption of maximal data using the so-called conformal Ansatz.

$$A^0\partial_\tau\phi + A^\alpha\partial_\alpha\phi = B(\Gamma_{ABCD})\phi.$$

Some properties of the evolution system

- The matrix associated to the ∂_τ term in the Bianchi propagation equations is given by:

$$A^0 = \sqrt{2}\text{diag}(1-\tau, 1, 1, 1, 1+\tau).$$

  - Thus, the equations degenerate at the sets where null infinity touches spatial infinity:

$$I^\pm = \{\rho = 0, \tau = \pm 1\}$$

  - Standard methods of symmetric hyperbolic systems cannot be used to analyse the equations near $I^\pm$.
- The cylinder at spatial infinity is a total characteristic ($A^\rho|_I = 0$) of the propagation equations:
  - The equations reduce to an interior system on I.

Methodological guideline:

"Existence and other properties of singular equations depend strongly on their algebraic structure."

Behaviour near the critical sets

General strategy:

Investigate the behaviour of the conformal field equations near the critical sets using asymptotic expansions.

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The asymptotic expansions:

- Assume that the field quantities admit the following *Taylor like expansions:*

$$v = \sum_{p\geq 0} \frac{1}{p!} v^{(p)}(\tau,\theta,\varphi)\rho^p, \qquad \phi = \sum_{p\geq 0} \frac{1}{p!} \phi^{(p)}(\tau,\theta,\varphi)\rho^p,$$

  where

$$v^{(p)}(\tau,\theta,\varphi) = \partial_\rho^p v|_I, \quad \phi^{(p)}(\tau,\theta,\varphi) = \partial_\rho^p \phi|_I.$$

- In order to determine the coefficients $v_j^{(p)}$ and $\phi_j^{(p)}$ exploit the fact that the cylinder I is a total characteristic of the conformal field equations.

A hierarchy of interior equations

The interior equations:

Obtained by differentiating the propagation equations with respect to ρ and evaluating on I renders a hierarchy of them.

$$\begin{aligned}\partial_\tau v^{(p)} = Kv^{(p)} &+ Q(v^{(0)}, v^{(p)}) + Q(v^{(p)}, v^{(0)}) \\ &+ \sum_{j=1}^{p-1} \Big(Q(v^{(j)}, v^{(p-j)}) + L^{(j)} \phi^{(p-j)} \Big) + L^{(p)} \phi^{(0)},\end{aligned}$$

$$\begin{aligned}A^{0,(0)} \partial_\tau \phi^{(p)} + A^{C,(p)} \partial_C \phi^{(p)} = B(\Gamma^{(0)}_{ABCD}) \phi^{(p)} \\ + \sum_{j=1}^{p} \begin{pmatrix} p \\ j \end{pmatrix} \Big(B(\Gamma^{(j)}_{ABCD}) \phi^{(p-j)} - A^{\mu,(j)} \partial_\mu \phi^{(p-j)} \Big),\end{aligned}$$

- The equations can be solved recursively —the equations are linear and decoupled.
- The coefficients $v_j^{(p)}$ and $\phi_j^{(p)}$ are completely determined by the expansions of the initial data on $\mathcal{S}$ near spatial infinity.

Solving the transport equations

Summary of the method:

- Expand $\upsilon^{(p)}$ and $\phi^{(p)}$ in terms of spherical harmonics:

$$\phi_j^{(p)} = \sum_{l=|2-j|}^{p} \sum_{m=-l}^{l} a_{j,p;l,m}(\tau)\,{}_sY_{lm}\rho^p.$$

- Analyse a system of ODE's for the τ dependent coefficients.

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- The transport equations for the υ-quantities are integrated without problems.
- In order to analyse in detail the structure of the solutions of the Bianchi propagation system, use the Bianchi constraints to obtain a reduced system.

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Computer algebra:

- Because of its algorithmic nature and of the linearity of the equations, the procedure can be appropriately implemented on a computer algebra system —e.g. `Maple V`. In some sense, the only limits are the hardware resources at hand!

The reduced system:

For a given multi-index $\alpha = (p, l, m)$, the reduced 2×2 system is of the form

$$y'_\alpha(\tau) = C_\alpha(\tau) y_\alpha(\tau) + b_\alpha(\tau)$$

with $C_\alpha(\tau)$ a 2×2 matrix valued function, and $y_\alpha(\tau)$ and $b_\alpha(\tau)$ 2-component vectors.

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Remarks:

- From the solutions to this system, one can calculate the expansion coefficients of $\phi^{(p)}$ and $\upsilon^{(p+1)}$.
- For given order p the vector $b_\alpha(\tau)$ is calculated from the lower order $\upsilon^{(q)}$ and the $\phi^{(q)}$'s. The matrices $C_\alpha(\tau)$ are independent of the lower order solutions —i.e. universal.

Solutions to the reduced system:

They can be written in the form

$$y_\alpha(\tau) = X_\alpha(\tau)X_\alpha(0)^{-1}y_\alpha(0) + X_\alpha(\tau)\int_0^\tau X_\alpha(s)^{-1}b_\alpha(s)ds,$$

where $X_\alpha(\tau)$ denotes a fundamental matrix of the system.

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The fundamental matrices:

- The fundamental matrices are known explicitly in terms of special functions.
- One can show that

  $$\det X = f(\tau)(1-\tau^2)^{p-2}$$

  with $f(\tau) \neq 0$ for $\tau \in [-1, 1]$.
  - There is, at least potentially, the possibility of solutions which are non-smooth at $\tau = \pm 1$.

More on the fundamental matrices

Regular fundamental matrices:

Given the multi-index $\alpha = (p, l, m)$, if $0 \leq l \leq p-1$, the fundamental matrices are expressible in terms of Jacobi polynomials.

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Singular fundamental matrices:

Given $\alpha = (p, l, m)$, if $l = p$ the fundamental matrices contain products of polynomials and $\ln(1 \pm \tau)$.

- The fundamental matrices are non-smooth!

A regularity condition

Can the effects of a singular fundamental matrix be avoided?

- Recall that the solutions to the reduced system are given by

$$y_\alpha(\tau) = X_\alpha(\tau)X_\alpha(0)^{-1}y_\alpha(0) + X_\alpha(\tau)\int_0^\tau X_\alpha(s)^{-1}b_\alpha(s)ds.$$

- The above are regular at the critical points if and only if

$$a_{0,p;p,m}(0) = a_{4,p;p,m}(0), \quad -p \leq m \leq p.$$

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Time symmetric data:

In this case the condition can be reformulated as

$$\mathfrak{C}(D_{k_p}\cdots D_{k_1}B_{lj})(i) = 0,$$

with B_{lj} the Coton-Bach tensor of the initial metric, $\mathfrak{C}$ denotes the trace-free part. It only involves freely specifiable data!

Are there any other singularities at the critical sets?

Does the regularity condition get hold of all the singularities at the critical sets?

- Back to the integral representation of the solutions to the reduced system:

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Going beyond the lazy man:

- Any further analysis requires the consideration of specific classes of initial data!

Outline

Time symmetric conformally flat initial data

The initial data:

- Work with a compactified initial hypersurface $\mathcal{S}$. One has that

$$\tilde{h}_{ij} = \vartheta^4 \delta_{ij}, \quad \chi_{ij} = 0.$$

- The Hamiltonian constraint implies the equation

$$\Delta \vartheta = \delta(i)$$

- In a neighbourhood of infinity the conformal factor can be written as:

$$\vartheta = 1/\rho + W,$$

with

$$W = \frac{1}{2}M + \sum_{l=2}^{\infty} \sum_{m=-l}^{l} w_{l,m} Y_{lm} \rho^l$$

Schwarzschildean data:

- An initial data set will be said to be Schwarzschildean to order p if the summation with respect to the index l starts at $l = p + 1$.

Solving the transport equations for the choice of data

How many spherical harmonic sectors to be solved?

At order p there are $p(p+2)$ sectors to be solved corresponding to

$$Y_{00}, \quad Y_{1m_1}, \quad Y_{2m_2}, \cdots, Y_{pm_p},$$

where

$$-p \leq m_p \leq p.$$

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Remarks:

- The number of sectors to be solved is a consequence of the choice of initial data.
- The sectors with $l = 0, 1$ do not give rise to singular solutions.
- The assumption of axial symmetry reduces in a significant manner the number of sectors to be analysed but still contains the essential complications of the general case.

Results from explicit calculations

Logarithmic singularities:

Explicit calculations up to order $p = 9$ have shown that for time symmetric, conformally flat data the coefficients $\phi^{(p)}$ develop logarithmic singularities at the critical points $I^{\pm}$, unless:

- the data is Schwarzschildean up to order $p = 6$.

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Expansion of W:

One has

$$w_{l,m} = 0 \quad l = 2, 3, 4, 5 \quad m = -l, \ldots l,$$

that is,

$$W = \frac{1}{2}M + \sum_{l=7}^{\infty} \sum_{m=-l}^{l} w_{l,m} Y_{l,m} \rho^{l}.$$

The big question:

Can one infer and prove general patterns from the computer algebra calculations?

Conjecture

Assume that one has an initial data set which is Schwarzschildean up to order p:

$$W = \frac{1}{2}M + \sum_{m=-l}^{l} w_{p+1,l} Y_{p+1,l} \rho^{p+1} + \cdots .$$

Then there is a $p_* > p$ such that the solutions of the reduced system

$$y_\alpha(\tau) = X_\alpha(\tau) X_\alpha(0)^{-1} y_\alpha(0) + X_\alpha(\tau) \int_0^\tau X_\alpha(s)^{-1} b_\alpha(s) ds, \quad \alpha = (p', l, m)$$

are logarithm-free for $p' < p_*$, but for $p' = p_*$ there are sectors which contain logarithmic divergences.

- *The solutions belong to the ring generated by $\ln(1 \pm \tau)$ and polynomials in τ. These logarithmic divergences do not arise if and only if*

$$w_{p+1,m} = 0, \quad m = -p-1, \ldots, p+1,$$

—i.e the data is Scharzschildean to order $p+1$.

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A simpler model problem

Problems proving the conjecture:

- The large number of propagation equations (50) complicates a direct attack on the problem. It would be useful to have a simpler model problem.

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Possible alternatives:

- The most natural option would be to consider a spin-2 massless field over a Schwarzschild background:
  - However this problem is overdetermined (Buchdahl constraints)!
  - The constraint equations cannot be dropped as they are required to calculate the reduced system.
- The next best thing is to consider a spin-1 Maxwell field:
  - 3 evolution equations and 1 constraint.

Implementing the model problem:

- Transport equations for the Maxwell field at the cylinder at spatial infinity of the Schwarzschild spacetime can be introduced without major problems.

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The reduced system:

- As in the case of the CFE, the analysis can be reduced to the study of solutions of a (2×2) reduced system

$$y'_\alpha(\tau) = C_\alpha(\tau) y_\alpha(\tau) + b_\alpha(\tau)$$

with solutions of the form

$$y_\alpha(\tau) = X_\alpha(\tau) X_\alpha(0)^{-1} y_\alpha(0) + X_\alpha(\tau) \int_0^\tau X_\alpha(s)^{-1} b_\alpha(s) ds,$$

where

$$\det X = C(1-\tau^2)^{p-1}.$$

The initial data

Data expandable in powers of ρ:

Consider initial data for the Maxwell equations such that the first 2^p-polar terms appear at order p. That is,

$$\phi_0 = \sum_{p=0}^{\infty}\sum_{l=1}^{p}\sum_{m=-l}^{l}\frac{1}{p!}a_{0,p;l,m}(0)\,{}_1Y_{lm}\rho^p,$$

$$\phi_1 = \sum_{p=0}^{\infty}\sum_{l=1}^{p}\sum_{m=-l}^{l}\frac{1}{p!}a_{1,p;l,m}(0)Y_{lm}\rho^p,$$

$$\phi_2 = \sum_{p=0}^{\infty}\sum_{l=1}^{p}\sum_{m=-l}^{l}\frac{1}{p!}a_{2,p;l,m}(0)\,{}_{-1}Y_{lm}\rho^p.$$

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A regularity condition:

The development has logarithmic singularities at the critical points unless

$$a_{0,p;p,m}(0) = a_{2,p;p,m}(0), \quad -p \leq m \leq p.$$

On the regularity conditions

Regularity condition for Maxwell data:

The development has logarithmic singularities at the critical points unless

$$a_{0,p;p,m}(0) = a_{2,p;p,m}(0), \quad -p \leq m \leq p.$$

Regularity condition for time symmetric data for the conformal Einstein field equations:

The analogous of the previous condition for the time symmetric initial data for the Einstein equations

$$a_{0,p;p,m}(0) = a_{4,p;p,m}(0), \quad -p \leq m \leq p.$$

can be rewritten as

$$\mathfrak{C}(D_{k_p} \cdots D_{k_1} B_{lj})(i) = 0,$$

with B_{lj} the Coton-Bach tensor of the initial metric, $\mathfrak{C}$ denotes the trace-free part.

Question:

Are the singularities precluded by this regularity condition the only ones that can arise?

Answer: no!

Theorem (JAVK 07)

Assuming that the initial data satisfies the regularity condition, the solutions to the reduced system for the sectors Y_{pm} are:

- *polynomial in τ for the orders ρ^p and ρ^{p+1};*
- *belong to the ring generated by polynomials in τ and $\ln(1 \pm \tau)$ for order ρ^{p+2}.*

Outline

The conjecture for the conformal Einstein field equations:

Conjecture

Assume that one has an initial data set which is Schwarzschildean up to order p*:*

$$W = \frac{1}{2}M + \sum_{m=-l}^{l} w_{p+1,l} Y_{p+1,l} \rho^{p+1} + \cdots .$$

Then there is a p_* *such that the solutions of the reduced system*

$$y_\alpha(\tau) = X_\alpha(\tau) X_\alpha(0)^{-1} y_\alpha(0) + X_\alpha(\tau) \int_0^\tau X_\alpha(s)^{-1} b_\alpha(s) ds, \quad \alpha = (p', l, m)$$

are logarithm-free for $p' < p_*$*, but for* $p' = p_*$ *there are sectors which contain logarithmic divergences.*

- *The solutions belong to the ring generated by* $\ln(1 \pm \tau)$ *and polynomials in* τ*. These logarithmic divergences do not arise if and only if*

  $$w_{p+1,m} = 0, \quad m = -p-1, \ldots, p+1,$$

  —i.e. the data is Scharzschildean to order $p+1$*.*

A proof?

Tentative strategy:

- As in the case of the Maxwell field on the Schwarzschild spacetime, attempt to construct a proof by a (more or less) direct and explicit calculation.
- Look at the behaviour of the solutions associated with the harmonics $Y_{p+1,m}$.

Integration for the lower orders

The data:

The data is Schwarzschildean to order p

$$W = \frac{1}{2}M + \sum_{m=-l}^{l} w_{p+1,l} Y_{p+1,l} \rho^{p+1} + \cdots .$$

The result:

For $p' < p+1$ the integration of the transport equations is identical to the case of the Schwarzschild spacetime up to the corresponding order.

What happens at order $p+1$?

Result:

- The υ-transport equations can be integrated trivially to give

$$\upsilon^{(p+1)} = 0$$

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- On the other hand

$$y_\alpha(\tau) = X_\alpha(\tau)X_\alpha(0)^{-1}y_\alpha(0) + X_\alpha(\tau)\int_0^\tau X_\alpha(s)^{-1}b_\alpha(s)ds,$$

  In this case, $b_\alpha = 0$ for the relevant multi-indices (i.e. the sectors $Y_{p+1,m}$). Consequently, the solutions $y_\alpha(\tau)$ are trivially polynomial in τ.

What happens at order $p+2$?

Result

- Using the results of the order $p+1$, the solutions to the υ-transport equations can be explicitly integrated in terms of hypergeometric functions.
- The product $X_\alpha(s)^{-1}b_\alpha(s)$ in the integral

$$y_\alpha(\tau) = X_\alpha(\tau)X_\alpha(0)^{-1}y_\alpha(0) + X_\alpha(\tau)\int_0^\tau X_\alpha(s)^{-1}b_\alpha(s)ds,$$

contains rational expressions of the form

$$\frac{P_\pm(\tau)}{(1\pm\tau)^3} = Q_\pm(\tau) + \frac{a_1}{(1\pm\tau)} + \frac{a_2}{(1\pm\tau)^2} + \frac{a_3}{(1\pm\tau)^3}.$$

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Non-trivially:

- A remarkable cancellation occurs!!

$$a_1 = 0.$$

No $(1\pm\tau)^{-1}$ terms are present. There are no logarithms at this order!

What happens at order $p+3$?

Result:

- Again, one can find explicit expressions for the solutions to the order $p+3$ υ-transport equations.
- One finds that the product $X_\alpha(s)^{-1}b_\alpha(s)$ in the integral

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$$\frac{P_\pm(\tau)}{(1\pm\tau)^{p+4}} = Q_\pm(\tau) + \frac{a_1}{(1\pm\tau)} + \frac{a_2}{(1\pm\tau)^2} + \cdots + \frac{a_{p+4}}{(1\pm\tau)^{p+4}}$$

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Non-trivial result:

- A lengthy inductive argument shows that

$$a_1 = 0.$$

- Again, there are no logarithms at this order!

Again with the conjecture...

Conjecture

Assume that one has an initial data set which is Schwarzschildean up to order p:

$$W = \frac{1}{2}M + \sum_{m=-l}^{l} w_{p+1,l} Y_{p+1,l} \rho^{p+1} + \cdots .$$

Then there is a p_* such that the solutions of the reduced system

$$y_\alpha(\tau) = X_\alpha(\tau) X_\alpha(0)^{-1} y_\alpha(0) + X_\alpha(\tau) \int_0^\tau X_\alpha(s)^{-1} b_\alpha(s) ds, \quad \alpha = (p', l, m)$$

are logarithm-free for $p' < p_*$, but for $p' = p_*$ there are sectors which contain logarithmic divergences.

- *The solutions belong to the ring generated by $\ln(1 \pm \tau)$ and polynomials in τ. These logarithmic divergences do not arise if and only if*

$$w_{p+1,m} = 0, \quad m = -p-1, \ldots, p+1,$$

—i.e the data is Schwarzschildean to order $p+1$.

What does one expect to happen at order $p+4$?

Insights:

- The analysis of particular explicit examples indicate that no cancellations occur at this order, so that the rational expressions in

$$y_\alpha(\tau) = X_\alpha(\tau)X_\alpha(0)^{-1}y_\alpha(0) + X_\alpha(\tau)\int_0^\tau X_\alpha(s)^{-1}b_\alpha(s)ds,$$

do produce logarithms, and that these logarithms are not present if and only if

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  do produce logarithms, and that these logarithms are not present if and only if

$$w_{p+1,m} = 0, \quad m = -p-1, \ldots, p+1.$$

- In the language of the conjecture one would have that

$$p_* = p+4.$$

Still to be addressed...

One would like to understand the following:

- Why are the expansions more regular than what *a priori* one would expect?
- The remarkable cancellations occurring in the expansions at order $p+2$ and $p+3$ show that there is some crucial structure which has not yet been identified/exploited.

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More general classes of data:

- Can one generalise the discussion to more general classes of data?
  - Probably yes but the calculations will be more involved. What part of the information available is the really essential one?

Some bibliography:

- JAVK. *A new class of obstructions to the smoothness of null infinity.* Comm. Math. Phys. **244**, 133-156 (2004).
- JAVK. *The Maxwell field on the Schwarzschild spacetime: behaviour near spatial infinity.* Proc. Roy. Soc. Lond. A **463**, 2609-2630 (2007).