Characterisation of initial data sets for the Einstein equations, reviewed

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Outline

Characterisation of initial data sets?

Assumptions:

1. Let $(\mathcal{S}, h_{ij}, K_{ij})$ satisfy the vacuum Einstein constraints

$$r + K^2 - K_{ij}K^{ij} = 0,$$
$$D^j K_{ij} - D_i K = 0.$$

2. Let $(\mathcal{M}, g_{\mu\nu})$ be a spacetime satisfying the vacuum Einstein field equations,

$$R_{\mu\nu} = 0.$$

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2. Let $(\mathcal{M}, g_{\mu\nu})$ be a spacetime satisfying the vacuum Einstein field equations,

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Question:

Under which conditions can one regard $(\mathcal{S}, h_{ij}, K_{ij})$ as a hypersurface in $(\mathcal{M}, g_{\mu\nu})$?

Characterisation of initial data sets?(II)

More precisely:

Given $(\mathcal{S}, h_{ij}, K_{ij})$ and $(\mathcal{M}, g_{\mu\nu})$, under which conditions there is an injective

$$\phi : \mathcal{S} \rightarrow \mathcal{M},$$

with associated unit normal $n_\mu \in T(\phi(\mathcal{S}))$, $g^{\mu\nu} n_\mu n_\nu = -1$, such that given

$$h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$$

one has

$$\begin{aligned} h_{ij} &= \partial_i \phi^\mu \partial_j \phi^\nu g_{\mu\nu}, \quad K_{ij} = \partial_i \phi^\mu \partial_j \phi^\nu (\mathcal{L}_n h)_{\mu\nu}. \\ &= \partial_i \phi^\mu \partial_j \phi^\nu h_{\mu\nu}, \end{aligned}$$

Remarks

Exact solutions:

The spacetime $(\mathcal{M}, g_{\mu\nu})$ will be assumed to be an exact solution. In this talk attention will be restricted to:

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An inverse problem:

The characterisation of initial data sets can be regarded as an inverse problem to the Cauchy problem in GR.

Minkowski spacetime

Local solution to the problem of characterisation of initial data:

The pair (h_{ij}, K_{ij}) of symmetric tensors corresponds (locally) to the first and second fundamental form of a slice $\mathcal{S}$ in Minkowski spacetime if and only if

$$D_{[i}K_{j]l} = 0,$$
$$r_{ijkl} = -2K_{k[i}K_{j]l}.$$

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Potential applications

Numerical Relativity:

In numerical simulations of dynamical black hole spacetimes (which make use of 3+1 formulations of GR) , one expects that the final state will be close to Kerr/Schwarzschild.

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Non-linear stability of Kerr:

Again, one expects that a dynamical spacetime which is the development of data close to Kerr data will have the same asymptotic structure as Kerr.

- What does it mean to be close to Kerr? (either at the data or spacetime level)

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Construction of initial data sets:

There is a number of conjectures about the existence/non-existence of certain types of hypersurfaces in Kerr/Schwarzschild. An invariant characterisation of data could be of use to construct the required data or to conclude that they do not exist.

- How to combine the standard methods to construct initial data with the extra conditions arising from the characterisations?

Some connections

Loose idea:

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Equivalence problem:

- Given two metrics $g_{\mu\nu}$ and $g'_{\mu\nu}$ on a Riemannian manifold, when are they (locally) isometric?
- Solved by Cartan (1946), requires at most the calculation of derivatives of tenth order of the Riemann tensor.

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Invariant characterisations of spacetimes:

- Given a spacetime $(\mathcal{M}, g_{\mu\nu})$ is there a set of algebraically independent invariants $\{\mathcal{I}_1, \mathcal{I}_2, \ldots\}$ of $g_{\mu\nu}$ (locally) characterising the spacetime?

How to attack the problem? A methodology.

Necessary conditions:

Start from an invariant characterisation of the required spacetime, $(\mathcal{M}, g_{\mu\nu})$, and perform a 3+1 decomposition of the characterisation.

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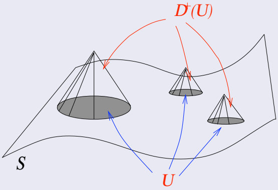
Sufficient conditions:

To obtain sufficient conditions, construct (hyperbolic) evolution equations for the invariants of the spacetime characterisation.

- Find under which conditions one can prove that if the invariants vanish on a Cauchy hypersurface $\mathcal{S} \subset \mathcal{M}$ then they vanish also at later times.



The Petrov classification (I)

A primer:

- An algebraic characterisation of the Weyl tensor of the spacetime, independent of any special coordinate system.
- It is based on the eigenvalue problem:

$$\frac{1}{2}\mathcal{C}_{\mu\nu}{}^{\lambda\rho}\mathcal{X}_{\lambda\rho} = \lambda\mathcal{X}_{\mu\nu},$$

where the self dual of the Weyl tensor $C_{\mu\nu\lambda\rho}$ is given by

$$\mathcal{C}_{\mu\nu\lambda\rho} \equiv C_{\mu\nu\lambda\rho} + \frac{i}{2}\epsilon_{\mu\nu\sigma\tau}C^{\sigma\tau}{}_{\lambda\rho},$$

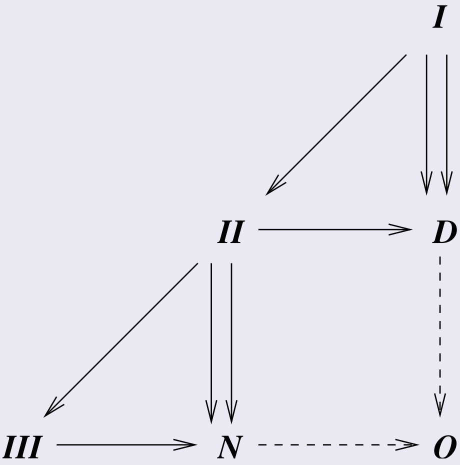
and

$$\mathcal{X}_{\mu\nu} \equiv X_{\mu\nu} + \frac{i}{2}\epsilon_{\mu\nu\sigma\tau}X^{\sigma\tau},$$

where $X_{\mu\nu} = X_{[\mu\nu]}$ is a bivector.

The Petrov classification (II)

The Penrose diagram:



Outline

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The Zakharov property

Theorem (Zakharov 1965, 1970, 1972)

Vacuum fields satisfying the relation

$$\nabla^\sigma \nabla_\sigma C_{\mu\nu\lambda\rho} = \alpha C_{\mu\nu\lambda\rho},$$

are either type N ($\alpha = 0$) or type D ($\alpha \neq 0$), where α is a function.

Observation:

- All vacuum Petrov type D spacetimes are known (Kinnersley, 1967).

Which spacetimes satisfy the Zakharov property?

Table of type D spacetimes:

$\rho \neq 0$	case I (NUT metrics including Schwarzschild)	only if $l = 0$
	case II.A (Kerr-NUT)	no
	case II.B	no
	case II.C	no
	case II.D	no
	case II.E	no
	case II.F	no
	case III.A (C-metric)	yes
	case III.B (twisting C-metric)	no
$\rho = 0$	case A	yes
	case B	no

Weyl-like tensors (Weyl-candidates)

Another observation:

The tensor $W_{\mu\nu\lambda\rho} \equiv \nabla^\sigma \nabla_\sigma C_{\mu\nu\lambda\rho}$ is a Weyl candidate —that is, it is tracefree and has the same symmetries as the Weyl tensor.

Electric and magnetic parts:

Given a Weyl-candidate $W_{\mu\nu\lambda\rho}$ one can construct its n-electric and n-magnetic parts:

$$\mathcal{E}[W]_{\tau\sigma} = W_{\mu\nu\lambda\rho} h^\mu{}_\tau n^\nu h^\lambda{}_\sigma n^\rho, \quad \mathcal{B}[W]_{\tau\sigma} = W^*_{\mu\nu\lambda\rho} h^\mu{}_\tau n^\nu h^\lambda{}_\sigma n^\rho.$$

Then one has that

$$W_{\mu\nu\lambda\rho} = 2\Big(l_{\mu[\lambda}\mathcal{E}[W]_{\rho]\nu} - l_{\nu[\lambda}\mathcal{E}[W]_{\rho]\mu} - n_{[\lambda}\mathcal{B}[W]_{\rho]\tau}\epsilon^\tau{}_{\mu\nu} - n_{[\mu}\mathcal{B}[W]_{\nu]\tau}\epsilon^\tau{}_{\lambda\rho}\Big),$$

with

$$l_{\mu\nu} = n_{\mu\nu} + h_{\mu\nu}.$$

The electric and magnetic parts of $\nabla^\sigma\nabla_\sigma C_{\mu\nu\lambda\rho}$

An identity for vacuum spacetimes:

$$\nabla^\sigma\nabla_\sigma C_{\mu\nu\lambda\rho} = C^\sigma{}_{\tau\mu\nu}C^\tau{}_{\sigma\lambda\rho} + 2(C^\sigma{}_{\mu\rho\tau}C^\tau{}_{\lambda\nu\sigma} - C^\sigma{}_{\nu\rho\tau}C^\tau{}_{\lambda\mu\sigma}).$$

Final result:

A lenghty calculation renders the neat expressions:

$$\mathcal{E}[W]_{\mu\nu} = 6\left(E_{\mu\sigma}E^\sigma{}_\nu - \frac{1}{3}h_{\mu\nu}E^{\sigma\tau}E_{\sigma\tau}\right) - 6\left(B_{\mu\sigma}B^\sigma{}_\nu - \frac{1}{3}h_{\mu\nu}B^{\sigma\tau}B_{\sigma\tau}\right),$$

$$\mathcal{B}[W]_{\mu\nu} = 12\left(E^\sigma{}_{(\mu}B_{\nu)\sigma} - \frac{1}{3}h_{\mu\nu}E_{\sigma\tau}B^{\sigma\tau}\right),$$

with

$$E_{\mu\nu} = \mathcal{E}[C]_{\mu\nu}, \quad B_{\mu\nu} = \mathcal{B}[C]_{\mu\nu}.$$

A first result

E_{ij} and B_{ij} in terms of initial data:

A direct consequence of the Codazzi equations is that one can write

$$E_{ij} = r_{ij} + KK_{ij} - K_{ik}K^k{}_j,$$
$$B_{ij} = \epsilon_{(i}{}^{kl} D_{|k}K_{l|j)}.$$

Theorem (JAVK, 2005)

Necessary conditions for an initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ *to be a Schwarzschild initial data set are:*

$$6\left(E_{ik}E^k{}_j - \frac{1}{3}h_{ij}E^{kl}E_{kl}\right) - 6\left(B_{ik}B^k{}_j - \frac{1}{3}h_{ij}B^{kl}B_{kl}\right) = \alpha E_{ij},$$
$$12\left(E^k{}_{(i}B_{j)k} - \frac{1}{3}h_{ij}E_{kl}B^{kl}\right) = \alpha B_{ij},$$

where

$$\alpha = 12E_i{}^k B_{jk}E^{ij}/E_{ij}B^{ij}.$$

Sufficient conditions?

How to single out Schwarzschild:

In order to recover Schwarzschild out of the set of spacetimes satisfying the Zakharov condition one could make use of global conditions:

- regular slices,
- asymptotically Euclidean slices.

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Propagation of the Zakharov property:

To conclude the argument one would have to analyse the propagation of the Zakharov property.

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- The tensor

$$Z_{\mu\nu\lambda\rho} \equiv \nabla^\sigma \nabla_\sigma C_{\mu\nu\lambda\rho} - \alpha C_{\mu\nu\lambda\rho},$$

  is a Weyl candidate. The "superenergy" techniques of Bonilla & Senovilla (1997) can be used to discuss the causal propagation of $Z_{\mu\nu\lambda\rho}$:
  - If $Z_{\mu\nu\lambda\rho} = 0$ on $\mathcal{S}$ then $Z_{\mu\nu\lambda\rho} = 0$ on $D^+(\mathcal{S})$.

Outline

A characterisation in terms of the Weyl tensor

Theorem (Ferrando & Sáez, 1998)

There is a set of scalars and tensors $\{\rho, \alpha, S_{\mu\nu\lambda\rho}, P_{\mu\nu}, Q_{\mu\nu}\}$ *constructed out of contractions of* $C_{\mu\nu\lambda\rho}$ *and* $\nabla_\sigma C_{\mu\nu\lambda\rho}$ *such that if*

$$\rho \equiv \left(\frac{1}{96} C_{\mu\nu}{}^{\lambda\rho} C_{\lambda\rho}{}^{\pi\sigma} C_{\pi\sigma}{}^{\mu\nu}\right)^{1/3}, \qquad \rho \neq 0,$$

$$\alpha \equiv \frac{1}{9\rho^2} g^{\mu\nu} \nabla_\mu \rho \nabla_\nu \rho + 2\rho, \qquad \alpha > 0,$$

$$S_{\mu\nu\lambda\sigma} \equiv \frac{1}{3\rho^2} \left(C_{\mu\nu\lambda\rho} + \rho(g_{\mu\nu} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\sigma})\right), \qquad S_{\mu\nu\kappa\pi} S^{\kappa\pi}{}_{\lambda\rho} - 2S_{\mu\nu\lambda\rho} = 0,$$

$$P_{\mu\nu} \equiv C^*_{\lambda\mu\sigma\nu} \nabla^\lambda \rho \nabla^\sigma \rho, \qquad P_{\mu\nu} = 0,$$

$$Q_{\mu\nu} \equiv S_{\lambda\mu\sigma\nu} \nabla^\lambda \rho \nabla^\sigma \rho, \qquad 2Q_{\mu\nu} v^\mu v^\nu + Q^\nu{}_\nu \leq 0,$$

where v^μ *is an arbitrary timelike vector, then* $(\mathcal{M}, g_{\mu\nu})$ *is (locally) isometric to the Schwarzschild spacetime. Moreover,*

$$\xi^\mu = \frac{1}{\rho^{4/3}\sqrt{Q_{\pi\sigma} v^\pi v^\sigma}} Q^\mu{}_\nu v^\nu$$

A set of necessary conditions:

Theorem (A. García-Parrado & JAVK, 2006)

Let

$$\rho = \left(\frac{1}{2}B_i{}^j B^{il} E_{jl} - \frac{1}{6}E_{ij}E_l{}^j E^{il}\right)^{1/3},$$

$$P = -\frac{1}{2}E^{ij}K_{ij} - \rho K - \frac{1}{6\rho}\epsilon^{jk}{}_i\left(E^{il}D_k B_{lj} + B^{il}D_k E_{lj}\right),$$

$$P_i = \frac{1}{6\rho}(-B^{kl}D_i B_{kl} + E^{kl}D_i E_{kl}).$$

Necessary conditions for an initial data set $(\mathcal{S}, h_{ij}, K_{ij})$*, satisfying the Einstein vacuum constraints to be a Schwarzschild initial data set are:*

$$B_{ij} = -\frac{1}{\rho}(B_i{}^k E_{kj} + B_j{}^k E_{ki}) \qquad B_{ij}P^i P^j = 0,$$

$$E_{ij} = \frac{1}{\rho}(B_i{}^k B_{kj} - E_i{}^k E_{kj}) + 2\rho h_{ij}, \qquad E_{jk}\epsilon^k{}_{li}P^j P^l - PB_{ij}P^j = 0,$$

$$(P^2 + P^k P_k)B_{ij} + 2PE_{k(i}\epsilon^k{}_{j)l}P^l - 2P_{(i}B_{j)l}P^l = 0,$$

$$\frac{1}{3}P_k P^k - P^2 + \frac{2}{3}E_{kl}P^k P^l > 0, \qquad \frac{1}{9\rho^2}(P^k P_k - P^2) + 2\rho > 0.$$

Sufficiency?

Propagation:

In order to investigate the sufficiency of the conditions given in the previous proposition, one has to analyse their propagation on the development of $(\mathcal{S}, h_{ij}, K_{ij})$.

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Alternatively!

The characterisation by F & S yields an explicit formula for the timelike Killing vector of the Schwarzschild spacetime in terms, again, of concomitants of $C_{\mu\nu\lambda\rho}$.

- Decompose this expression to obtain a Killing initial data candidate (a KID candidate).

Killing Initial Data (KID) in brief

See e.g. Moncrief, Coll, 1975; Beig & Chruściel , 1996, 1997; Beig, Chruściel, Schoen 2005.

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The KID equations:

A $3+1$ decomposition of the Killing equation $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$, where one writes

$$\xi^\mu = Y n^\mu + Y^\mu, \quad n^\mu Y_\mu = 0,$$

renders the Killing initial data equations

$$D_{(i}Y_{j)} - YK_{ij} = 0,$$
$$D_i D_j Y - (\mathcal{L}_Y K)_{ij} = Y(r_{ij} + KK_{ij} - 2K_{il}K^l{}_j).$$

A pair (Y, Y^i) on a hypersurface $\mathcal{S}$ is called a KID if and only if it satisfies the above KID equations.

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One-to-one correspondence between KIDs and Killing vectors:

Given $(\mathcal{S}, h_{ij}, K_{ij})$ (vacuum) possessing a KID, then the development of the initial data has a Killing vector (modulo some regularity conditions) whose pull back to $\mathcal{S}$ coincides with the KID.

A KID candidate for Schwarzschild initial data

The $3+1$ decomposition of F&S Weyl concomitants formula for the timelike Killing vector renders:

$$Y \equiv \frac{\sqrt{|\rho P_i P^i - E_{ij} P^i P^j|}}{\rho^{11/6}\sqrt{3}},$$

$$Y^i \equiv \frac{-\rho P P^i + E^{ij} P_j P - B^{kl} P_l P_m \epsilon^{im}{}_k}{\rho^{11/6}\sqrt{3|\rho P_j P^j - E_{kl} P^k P^l|}}.$$

Conjecture

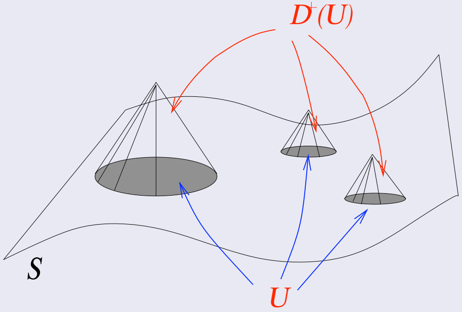
If $(\mathcal{S}, h_{ij}, K_{ij})$ satisfies the necessary conditions arising from the $3+1$ decomposition of F&S concomitant conditions, then the KID candidate is actually a KID.

- Proven for the time symmetric case ($K_{ij} = 0$) —AGP & JAVK.

An algorithmic procedure

Alternatively:

- If for the initial data set under consideration the KID candidate is indeed a KID, one can ensure that the necessary conditions are propagated in the domain of dependence $\mathcal{D}^+(\mathcal{U})$ of the parts of $\mathcal{S} \supset \mathcal{U}$ where the KID is timelike $Y^2 - Y^i Y_i < 0$.



- In this way one obtains a set of sufficient conditions.
- This is, in fact, an algorithmic procedure to verify whether an initial data set is a Schwarzschildean initial data.

Outline

1. Motivation
2. Characterisations of Schwarzschild data
   - The Zakharov property
   - An invariant characterisation of Schwarzschild
3. Characterisations of Petrov type D initial data
4. Characterisations of Kerr initial data
5. Perspectives

Killing spinors as a characterisation of type D spacetimes

Rationale:

- Characterisations in terms of the algebraic conditions determining the Petrov type (e.g. type D) are cumbersome.
- Can one find a way of using the Petrov type in an indirect manner?

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Killing spinors (cfr. Penrose & Rindler Vol.2)

- A Killing spinor is a totally symmetric spinor $\kappa_{(AB\cdots P)} = \kappa_{AB\cdots P}$ satisfying

$$\nabla_{Q'(Q}\kappa_{AB\cdots P)} = 0.$$

- Consider the particular cases where the spinor has either one or two indices

$$\nabla_{A'(A}\kappa_{B)} = 0, \quad \text{(Twistor equation)}$$
$$\nabla_{A'(A}\kappa_{BC)} = 0. \quad \text{(Valence-2 Killing spinor equation)}$$

Killing spinors and Petrov type

Restrictions on the curvature:

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The twistor and valence-2 Killing spinor equation are overdetermined —they impose strong conditions on the curvature of the spacetime.

Theorem (See Penrose & Rindler Vol. 2)

A spacetime admits a solution to the twistor equation if and only if it is of Petrov type N.

Theorem (Jeffryes 1984; AGP & JAVK, 2008)

Any Petrov type D spacetime admits a valence-2 Killing spinor. Conversely, if a spacetime admits a valence-2 Killing spinor then the spacetime is of Petrov type D (and the spinor is algebraically special) or the spacetime is of Petrov type N (and the Killing spinor is degenerate).

Some further properties of (valence-2) Killing spinors:

Theorem (Penrose & Walker, 1970; Penrose & Rindler Vol.2)

Any vacuum spacetime with a Killing spinor κ_{AB} has a (complex) Killing vector given by

$$\xi_{AA'} \equiv \nabla^{Q}{}_{A'}\kappa_{AQ},$$

so that

$$\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = 0.$$

Theorem

For the Kerr spacetime the Killing vector associated to κ_{AB} is degenerate (i.e. its real and imaginary parts are proportional).

Killing spinor initial data?

Mimic the discussion of KIDs:

- In order to obtain a "3+1" decomposition of the Killing spinor equations, consider space-spinors (or SU(2) spinors).
- Given a foliation of spacetime with normal τ_μ ($\tau_\mu \tau^\mu = 2$), contract suitably with $\tau_{AA'}$ to eliminate all the primed indices in the expressions.

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Space-spinors:

Decompose vectors using

$$\xi_{AA'} = \frac{1}{2}\tau_{AA'}\xi - \tau_{A'}{}^{C}\xi_{AC}$$

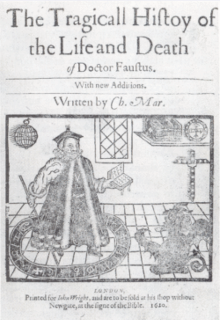
with

$$\xi \equiv \xi_{AA'}\xi^{AA'}, \quad \xi_{AB} \equiv \tau^{A'}{}_{(A}\xi_{B)A'},$$

$$\nabla_{AB} = \tau_{(B}{}^{A'}\nabla_{A)A'} \quad \text{(Sen connection)},$$

$$D_{AB}\kappa_C = \nabla_{AB}\kappa_C - K_{ABCQ}\kappa^Q,$$

(Levi-Civita connection).



Propagation of the twistor equation

Procedure:

- Construct a wave equation for

$$H_{A'AB} \equiv \nabla_{A'(A}\kappa_{B)}$$

which is homogeneous on $H_{A'AB}$ and $\nabla_{CC'}H_{A'AB}$. Namely,

$$\nabla^{EE'}\nabla_{EE'}H_{A'AB} = \Psi_{AB}{}^{PQ}H_{A'PQ}.$$

- Necessary and sufficient conditions for $H_{A'AB} \equiv 0$ on $D^+(\mathcal{S})$ are that

$$H_{A'AB} = 0, \quad \nabla_{EE'}H_{A'AB} = 0, \quad \text{on } \mathcal{S}.$$

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Need a space spinor decomposition of

$$\nabla_{A'(A}\kappa_{B)}(p) = 0, \quad \nabla_{EE'}\nabla_{A'(A}\kappa_{B)}(p) = 0, \quad p \in \mathcal{S}.$$

Theorem (AGP & JAVK, 2008)

If on $\mathcal{S}$ *there exists a* κ_A *such that*

$$\nabla_{(AB}\kappa_{C)} = 0,$$
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then $\mathcal{M}$ *has a Killing spinor associated to the above valence-1 Killing spinor initial data.*

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Warning!

The expressions above are deceptively simple as the Sen connection has been used.

Extending the discussion to valence-2 Killing spinors

Procedure:

Define

$$H_{A'ABC} \equiv \nabla_{A'(A}\kappa_{BC)},$$

$$S_{AA'BB'} \equiv \nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = -\frac{1}{4}\nabla^{Q}{}_{A'}H_{B'ABQ},$$

where $\xi_{AA'} = \nabla^{Q}{}_{A'}\kappa_{AQ}$. A calculation renders

$$\square H_{A'ABC} = 4\Psi_{(AB}{}^{PQ}H_{C)PQA'} + 8\nabla_{(A}{}^{Q'}S_{BC)Q'A'}.$$

Extending the discussion to valence-2 Killing spinors

Procedure:

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Observation:

Thus, one has to consider the propagation of the valence-2 Killing spinor equation in parallel to the propagation of the associated Killing equation!

Theorem (AGP & JAVK, 2008)

If on $\mathcal{S}$ *there is a* κ_{AB} *such that*

$$\nabla_{(AB}\kappa_{CD)} = 0, \quad \nabla_{(AC}\nabla_{B}{}^{D}\kappa_{F)D} = 0,$$

and

$$\xi \equiv D^{PQ}\kappa_{PQ}, \quad \xi_{BF} \equiv -\frac{1}{2}K^{DA}{}_{DA}\kappa_{BF} + \frac{3}{4}\kappa^{DA}K_{(BFDA)} + \frac{3}{2}D_{(F}{}^{D}\kappa_{B)D},$$

satisfy the KID equations, then on $\mathcal{M}$ *there is a valence-2 Killing spinor associated to the above Killing spinor data.*

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Observation:

The above result, together with the analogue result for valence-1 Killing spinors renders a characterisation of non-denegerate Petrov type D data.

Outline

The Mars-Simon tensor I

Ancilliary objects:

- Let $(\mathcal{M}, g_{\mu\nu})$ with a Killing vector ξ^μ. The associated *Killing form* is given by

$$F_{\mu\nu} \equiv \nabla_{[\mu}\xi_{\nu]} = \nabla_\mu \xi_\nu.$$

- The *self-dual Killing form* $\mathcal{F}_{\mu\nu}$ is given by

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + \mathrm{i} F^*_{\mu\nu}.$$

- The *Ernst 1-form* is given by

$$\sigma_\nu \equiv 2\xi^\mu \mathcal{F}_{\mu\nu} = -\nabla_\nu \lambda - \mathrm{i}\omega_\nu,$$

- Locally there exists a scalar field σ, the *Ernst potential* $\sigma_\mu = \nabla_\mu \sigma$.
- The metric in the space of self-dual forms which is given by

$$\mathcal{I}_{\mu\nu\lambda\rho} \equiv \frac{1}{2}\left(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda} + \mathrm{i}\eta_{\mu\nu\lambda\rho}\right).$$

The Mars-Simon-tensor II

Definition of the Mars-Simon tensor (Mars, 1999,2000; see also Simon, 1984):

In terms of the previous ancilliary quantities, one has

$$\mathcal{S}_{\mu\nu\lambda\rho} \equiv 2\overline{\mathcal{C}}_{\mu\nu\lambda\rho} + \frac{6}{1-\sigma}\left(\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho} - \frac{1}{6}\mathcal{F}^2\mathcal{I}_{\mu\nu\lambda\rho}\right),$$

with $\mathcal{C}_{\mu\nu\lambda\rho}$ the self-dual Weyl tensor.

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Remark:

In a vacuum spacetime admitting a Killing vector, the Mars-Simon tensor is determined up to a choice for the Ernst potential. We make a choice such that $\sigma \neq 1$ on $\mathcal{M}$.

A characterisation of the Kerr spacetime

Theorem (Mars, 2000)

Let $(\mathcal{M}, g_{\mu\nu})$ *be a smooth, vacuum spacetime with a Killing vector* ξ^μ*. Let* $\mathcal{F}_{\mu\nu}$ *denote the associated self-dual Killing form. If there is a non-vanishing real constant* M *such that the conditions*

$$\mathcal{F}^2 = -M^2(1-\sigma)^4,$$
$$\mathcal{S}_{\mu\nu\lambda\rho} = 0,$$

hold on a non-empty $\mathcal{N} \subset \mathcal{M}$ *then* $(\mathcal{N}, g_{\mu\nu})$ *is locally isometric to the Kerr spacetime.*

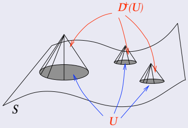
Some further properties of the Mars-Simon tensor (I)

Causal propagation of the Mars-Simon tensor:

A lenghty calculation (cfr. Klainerman & Ionescu, 2007) renders

$$\nabla_\alpha \mathcal{S}^\alpha{}_{\beta\gamma\delta} = \frac{1}{\sigma - 1}\left(4(\mathcal{F}_{[\delta}{}^{\rho}\mathcal{S}_{\gamma]\beta\alpha\rho} + \mathcal{F}_{\beta}{}^{\rho}\mathcal{S}_{\gamma\delta\alpha\rho}) + 2g_{\beta[\gamma}\mathcal{S}_{\delta]\alpha\lambda\rho}\mathcal{F}^{\lambda\rho}\right)\xi^\alpha.$$

- Using the previous result and the techniques of Bonilla & Senovilla (1997), one can show that the Mars-Simon tensor popagates causally:
  - If the Mars-Simon tensor vanishes initially on an hypersurface $\mathcal{S}$, then it vanishes on $D^+(\mathcal{S})$.



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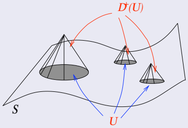

- This approach avoids considering a wave equation for the Mars-Simon tensor.
  - Such approach would impose extra conditions on the initial data set as not only the Mars-Simon tensor would be required to vanish on $\mathcal{S}$ but also its covariant derivative.

Some further properties of the Mars-Simon tensor (II)

Electric-magnetic parts decomposition:

The Mars-Simon tensor being Weyl-like and self-dual can be written as

$$\mathcal{S}_{\mu\nu\rho\lambda} = 2(l_{\nu[\lambda}\mathcal{T}_{\rho]\mu} + l_{\mu[\rho}\mathcal{T}_{\lambda]\nu} + \mathrm{i}\,\varepsilon_{\rho\lambda\alpha}n_{[\mu}\mathcal{T}_{\nu]}{}^{\alpha} + \mathrm{i}\,\varepsilon_{\mu\nu\alpha}n_{[\rho}\mathcal{T}_{\lambda]}{}^{\alpha}),$$

where

$$\mathcal{T}_{\mu\nu} \equiv \mathcal{S}_{\mu\rho\nu\lambda}n^{\rho}n^{\lambda}.$$

A theorem for Kerr data I

Theorem (Kerr initial data, AGP & JAVK, 2008)

Let $(\mathcal{S}, h_{ij}, K_{ij})$ *be a vacuum initial data set and assume that there exist two scalar fields* Y, σ, *a vector field* Y_j *and a real constant* M, *all defined on* $\mathcal{S}$, *fulfilling the following conditions*

$$\mathcal{E}_j\mathcal{E}^j = \frac{1}{4}M^2(1-\sigma)^4,$$

$$4\mathcal{E}^{ij}\overline{\mathcal{E}}_{ij} - 24\mathrm{Re}\left[\frac{\mathcal{E}^i\mathcal{E}^j\mathcal{E}_{ij}}{\sigma-1}\right] - \frac{3}{4}M^4|1-\sigma|^6 + 36\left|\frac{\mathcal{E}^i\overline{\mathcal{E}}_i}{\sigma-1}\right|^2 = 0,$$

$$D_j\sigma = 2YK_{jl}Y^l - 2\,\mathrm{i}\,\varepsilon_{jml}(K_k{}^lY^kY^m + Y^mD^lY) + 2YD_jY + 2Y^lD_{[l}Y_{j]},$$

with

$$\mathcal{E}_j \equiv K_{jk}Y^k + D_jY - \frac{1}{2}\mathrm{i}\,\varepsilon_{jkl}D^kY^l,$$

$$\mathcal{E}_{jk} \equiv \frac{1}{2}(E_{jk} - \mathrm{i}\,B_{jk}),$$

and Y, Y_j *solutions of the KID equations. Then the data development* $(\mathcal{M}, g_{\mu\nu})$ *of* $(\mathcal{S}, h_{ij}, K_{ij})$ *is locally isometric to an open subset of the Kerr spacetime.*

A theorem for Kerr data II

Specialisation to Schwarzschild

A specialisation of the previous result to the case of Schwarzschild data can be obtained by noting the following result:

Lemma (Carrasco & Mars, 2008)

Let $(\mathcal{S}, h_{ij}, K_{ij})$ be a vacuum initial data set and assume that there exist Y and Y_j on $\mathcal{S}$, satisfying

$$Y D_{[i}Y_{j]} + 2Y_{[i}D_{j]}Y + 2Y_{[i}K_{j]l}Y^l = 0$$
$$\tilde{Y}_{[i}D_j\tilde{Y}_{k]} = 0.$$

Then there exists an integrable Killing vector in the development of $(\mathcal{S}, h_{ij}, K_{ij})$.

A theorem for Kerr data II

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Then there exists an integrable Killing vector in the development of $(\mathcal{S}, h_{ij}, K_{ij})$.

Remark:

An explicit formula for a KID candidate is available in this case.

Outline

A KID candidate for Kerr data

Question:

Can one obtain an explicit expression for the timelike Killing vector of the Kerr in terms of the Weyl tensor and derived objects?

- If so, one could obtain a formula for a KID candidate for Kerr data.

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Remark:

The desired expression may already be implicit in some further analysis by Ferrando & Sáez of type D spacetimes.

Invariants for static data

Theorem (Dain, 2004)

There is a geometric invariant measuring the departure of time symmetric, asymptotically flat data from the static regime. If the invariant vanishes then the data is exactly static.

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Ancilliary objects:

Consider the constraint map

$$\Phi \begin{pmatrix} h_{ij} \\ K_{ij} \end{pmatrix} = \begin{pmatrix} r + K^2 - K_{ij}K^j \\ -D^j K_{ij} + D_i K \end{pmatrix}.$$

One can compute the linearisation of Φ evaluated at (h_{ij}, K_{ij})

$$D\Phi \begin{pmatrix} \gamma_{ij} \\ Q_{ij} \end{pmatrix} = \begin{pmatrix} D^i D^j \gamma_{ij} - r_{ij}\gamma^{ij} - \Delta\gamma + H \\ -D^j Q_{ij} + D_i Q - F_i \end{pmatrix},$$

and its formal adjoint

$$D\Phi^* \begin{pmatrix} \eta \\ X^i \end{pmatrix} = \begin{pmatrix} D_i D_j \eta - \eta r_{ij} - \Delta\eta h_{ij} + H_{ij} \\ D_{(i}X_{j)} - D^k X_k h_{ij} + F_{ij} \end{pmatrix}$$

KIDS again:

The elements of the Kernel of $D\Phi^*$ are the KIDs.

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Some more ancilliary objects:

Define the operators (Bartnik)

$$\mathcal{P}\begin{pmatrix} \gamma_{jk} \\ q_{ijk} \end{pmatrix} = D\Phi \begin{pmatrix} \gamma_{jk} \\ -D^i q_{ijk} \end{pmatrix},$$

and its adjoint

$$\mathcal{P}^*\begin{pmatrix} \eta \\ X_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D_j \end{pmatrix} D\Phi^* \begin{pmatrix} \eta \\ X_i \end{pmatrix}$$

Approximate symmetries:

We say (η, X_i) is an approximate symmetry if it satisfies the equation

$$\mathcal{P}\mathcal{P}^* \begin{pmatrix} \eta \\ X^i \end{pmatrix} = 0,$$

and has the fall-off behaviour at infinity of the Killing vectors of the flat spacetime.

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and has the fall-off behaviour at infinity of the Killing vectors of the flat spacetime.

Observations:

- Every symmetry is also an approximate symmetry.
- Generic time symmetric data has approximate symmetries (proven by S. Dain). The non-time symmetric case has not been proved.
- One can uniquely associate each approximate symmetry with a symmetry in flat space.

Geometric invariants for Petrov type D data

Question:

Can one adapt the previous ideas to the operators associated to the various Killing spinor initial data sets?

Geometric invariants for Petrov type D data

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- Approximate Killing spinors?

A $3+1$ version of the equivalence problem

Question:

Can one in a useful and meaningful manner reformulate the Riemannian equivalence problem in terms of h_{ij} and K_{ij} and derived objects?

A golden oldie

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LXVII CORSO - VARENNA SUL LAGO DI COMO - VILLA MONASTERO - 28 Giugno - 10 Luglio 1976