Mathematical problems of General Relativity
Problem sheet 4

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1. Verify that given the conformal rescaling $h_{ij} = \vartheta^4 \bar{h}_{ij}$, the associated Christoffels symbols are related to each other via
$$\Gamma^i{}_{jk} = \bar{\Gamma}^i{}_{jk} + 2(\delta_j{}^i \partial_k \ln\vartheta + \delta_k{}^i \partial_j \ln\vartheta - \bar{h}_{jk}\bar{h}^{il}\partial_l \ln\vartheta).$$

2. Verify that from the transformation rule for the Ricci tensor under conformal rescalings
$$r_{ij} = \bar{r}_{ij} - 2(\bar{D}_i\bar{D}_j \ln\vartheta + \bar{h}_{ij}\bar{h}^{lm}\bar{D}_l\bar{D}_m \ln\vartheta) + 4(\bar{D}_i \ln\vartheta \bar{D}_j \ln\vartheta - \bar{h}_{ij}\bar{h}^{lm}\bar{D}_l \ln\vartheta \bar{D}_m \ln\vartheta).$$
it follows that the Ricci scalar transforms as
$$r = \vartheta^{-4}\bar{r} - 8\bar{\theta}^{-5}\bar{D}_k\bar{D}^k\vartheta.$$

3. Verify that under the conformal rescaling $h_{ij} = \vartheta^4 \bar{h}_{ij}$ the time symmetric Hamiltonian constraint $r = 0$ implies the Yamabe equation
$$\bar{D}_k\bar{D}^k\vartheta - \tfrac{1}{8}\bar{r}\vartheta = 0$$
What is the corresponding expression in the case of non-vanishing extrinsic curvature?

4. Verify that the conformal factor of the Brill-Lindquist initial data is indeed a solution to the Yamabe equation.

5. Let n_a denote the unit normal of an hypersurface $\mathcal{S}$, and let ξ^a denote a Killing vector on the spacetime $(\mathcal{M}, g_{ab})$. If $\xi^a n_a = 0$ (i.e. the Killing vector is orthogonal to $\mathcal{S}$) show that the Frobenius condition
$$\xi_{[a}\nabla_b\xi_{c]} = 0$$
is satisfied.

6. Show that in a stationary spacetime, and using adapted coordinates such that $\xi^a\partial_a = \partial_t$, the Killing vector condition $\mathcal{L}_\xi g_{ab} = 0$ together with the definitions of h_{ij} and K_{ij} imply that
$$\partial_t h_{ij} = \partial_t K_{ij} = 0.$$

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