Mathematical problems of General Relativity
Lecture 3

Juan A. Valiente Kroon

School of Mathematical Sciences
Queen Mary, University of London
j.a.valiente-kroon@qmul.ac.uk,

LTCC Course LMS
Outline

1. The 3 + 1 decomposition of General Relativity
   - The 3+1 form of the spacetime metric

2. A closer look at the constraint equations

3. Time independent solutions
Outline

1. The 3 + 1 decomposition of General Relativity
   - The 3+1 form of the spacetime metric

2. A closer look at the constraint equations

3. Time independent solutions
Adapted coordinates (I)

Remarks:

- The discussion of the evolution equations given in the previous section has been completely general.
- The only assumption that has been made about the spacetime is that it is globally hyperbolic so that a foliation and a corresponding time vector exist.
- The discussion of the $3+1$ can be further particularised by introducing adapted coordinates. In this section we briefly discuss how this can be done.
Adapted coordinates (I)

Remarks:

- The discussion of the evolution equations given in the previous section has been completely general.
- The only assumption that has been made about the spacetime is that it is globally hyperbolic so that a foliation and a corresponding time vector exist.
- The discussion of the $3+1$ can be further particularised by introducing adapted coordinates. In this section we briefly discuss how this can be done.

Choosing the time coordinate:

- Recall that the hypersurfaces of the foliation of a spacetime $(\mathcal{M}, g_{ab})$ can be given as the level surfaces of a time function $t$.
- We already have seen that $\nabla_a t^a = 1$. The latter combined with $\nabla_\mu t = (1, 0, 0, 0)$ readily imply that $t^\mu = (1, 0, 0, 0)$. Hence, the Lie derivative along the direction of $t^a$ is simply a partial derivative —that is,

$$\mathcal{L}_t = \partial_t.$$
Adapted coordinates (II)

The shift vector:

- From the previous discussion it follows that the spatial components of the unit normal must vanish —i.e. $n_\alpha = 0$.
- Since the contraction of spatial vectors with the normal must vanish, it follows that all components of spatial tensors with a contravariant index equal to zero must vanish.
- For the shift vector one has that $n_\mu \beta^\mu = n_0 \beta^0 = 0$ so that $\beta^\mu = (0, \beta_\alpha)$.
- Since one has that $t^a = \alpha n^a + \beta^a$, it follows then that
  \[ n^\mu = (\alpha^{-1}, -\alpha^{-1} \beta^\alpha). \]

- Moreover, from the normalisation condition $n_a n^a = -1$ one finds
  \[ n_\mu = (-\alpha, 0, 0, 0). \]
Adapted coordinates (III)

The 3-metric:

- Recalling that \( h_{ab} = g_{ab} + n_a n_b \) one concludes that
  \[
  h_{\alpha\beta} = g_{\alpha\beta}.
  \]

- In these **adapted coordinates** the 3-metrics of the hypersurfaces of the
  foliation are simply the spatial part of the spacetime metric \( g_{ab} \).

- Moreover, since the time components of spatial contravariant tensors have to
  vanish, one also has that \( h^{\mu 0} = 0 \).

- One concludes that one can write
  \[
  g^{\mu\nu} = h^{\mu\nu} - n^\mu n^\nu = \begin{pmatrix}
  -\alpha^{-2} & \alpha^{-2} \beta^\gamma \\
  \alpha^{-2} \beta^\delta & h_{\gamma\delta} - \alpha^{-2} \beta^\gamma \beta^\delta
  \end{pmatrix}.
  \]

- This last expression can be inverted to yield
  \[
  g_{\mu\nu} = \begin{pmatrix}
  -\alpha^2 + \beta_\gamma \beta^\gamma & \beta_\gamma \\
  \beta_\gamma & h_{\gamma\delta}
  \end{pmatrix},
  \]

  where \( \beta_\gamma \equiv h_{\gamma\delta} \beta^\delta \).
The 3+1 form of the metric

The line element in adapted coordinates:

- An alternative way of presenting the latter is via the line element

\[ g = -\alpha^2 dt^2 + h_{\gamma\delta}(\beta^\gamma dt + dx^\gamma)(\beta^\delta dt + dx^\delta). \]

- The latter is known as the 3+1 form of the spacetime metric.
The constraint and evolution equations in adapted coordinates

A summary:

- The constraint and ADM evolution equations can be written in adapted coordinates as

\[
\begin{align*}
\partial_t h_{ij} &= -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i, \\
\partial_t K_{ij} &= -D_i D_j \alpha + \alpha (r_{ij} - 2K_{ik}K^k_{\ j} + KK_{ij}) \\
&\quad + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k,
\end{align*}
\]
The 3+1 decomposition of General Relativity

An example: the Schwarzschild spacetime (I)

Isotropic coordinates:

- The metric Schwarzschild spacetime can be expressed in standard coordinates in terms of the line element

\[ g = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2. \]

- This form of the metric is not the best one for a 3+1 decomposition of the spacetime.

- Instead, introduce an isotropic radial coordinate \( \bar{r} \) via \( r = \bar{r} \left(1 + \frac{m}{2\bar{r}}\right)^2 \).

- In terms of the later one obtains the line element of the Schwarzschild spacetime in the form

\[ g = -\left(1 - \frac{m/2\bar{r}}{1 + m/2\bar{r}}\right)^2 + \left(1 + \frac{m}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2d\theta^2 + \bar{r}^2\sin^2\theta d\varphi). \]
The gauge functions:

The normal $\omega_a = \nabla_a t$ is then readily given by

$$\omega_\mu = (1, 0, 0, 0).$$

Thus, one readily reads the lapse function to be

$$\alpha = \frac{1 - m/2\bar{r}}{1 + m/2\bar{r}},$$

while the unit normal is

$$n^\mu = \frac{1 + m/2\bar{r}}{1 - m/2\bar{r}} (1, 0, 0, 0).$$

Also, the shift vanishes: $\beta^\alpha = 0.$
The intrinsic metric and the extrinsic curvature:

- The spatial metric is then

\[ h = \left(1 + \frac{m}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2 \theta d\varphi). \]

- Since \( \beta^i = 0 \) and \( h_{ij} \) is independent of time, one can readily find that the extrinsic curvature vanishes

\[ K_{ij} = 0. \]

- The isotropic form of the Schwarzschild readily yields a foliation of spacetime that follows the static symmetry of the spacetime.

- In this foliation, the intrinsic 3-metric of the leaves does not seem to evolve.

- Any other foliation not aligned with the static Killing vectors will give rise to a non-trivial evolution for both \( h_{ij} \) and \( K_{ij} \).
A closer look at the constraint equations

1. The $3+1$ decomposition of General Relativity
   - The $3+1$ form of the spacetime metric

2. A closer look at the constraint equations

3. Time independent solutions
The constraint equations:

- The Einstein field equations imply the following constraint equations on a (spatial) hypersurface $S$:
  
  \[ r + K^2 - K_{ij}K^{ij} = 0, \quad \text{Hamiltonian constraint} \]
  
  \[ D^iK_{ij} - D_jK = 0. \quad \text{Momentum constraint} \]

- These equations constraint the possible choices of pairs $(h_{ij}, K_{ij})$ corresponding to initial data to the Einstein field equations.

- They are intrinsic equations, that is, they only involve objects which are defined on the hypersurface $S$ without any further reference to the “bulk” of the spacetime $(M, g_{ab})$. 
The constraint equations — summary (II)

PDE properties:

- The Einstein constraint represent a highly coupled, highly non-linear system of equations for \((h_{ij}, K_{ij})\).
- The main difficulty in constructing an solution to the equations lies in the fact that the equations constitute an underdetermined system: one has 4 equations for 12 unknowns — the independent components of two symmetric spatial tensors.
- Even exploiting the coordinate freedom to “kill off” three components of the tensors, one is still left with 9 unknowns.
- There should be some freedom in the specification of data for the equations.
  - The task is to identify what this free data is!!!
Simplifying assumptions: time symmetry

Vanishing of the extrinsic curvature

- In order to render the problem manageable, we make a standard simplifying assumption and consider initial data sets for which $K_{ij} = 0$ everywhere on $S$.
- This class of initial data are called **time symmetric**.
- The reason for this is that if $K_{ij} = 0$ at $S$ then the evolution equations imply that
  \[
  \partial_t h_{ij} = 0, \quad \text{on} \quad S.
  \]
  This equation is invariant under the replacement $t \mapsto -t$.
- It follows that the resulting spacetime has a reflection symmetry with respect to the hypersurface $S$ which can be regarded as a **moment of time symmetry**.
The Hamiltonian constraint:

- If $K_{ij} = 0$ everywhere on $S$ then the momentum constraint is automatically solved, and the Hamiltonian constraint reduces to $r = 0$.

- That is, the initial 3-metric has to be such that its Ricci scalar —notice that this does not mean that the hypersurface is flat!

- The time symmetric Hamiltonian constraint regarded as an equation for $h_{ij}$ is highly non-linear.

- Moreover, one still has 6 unknowns and equation —even choosing coordinates, one still has 3 unknowns.
A closer look at the constraint equations

Conformal rescalings and the Yamabe problem

A strategy:

- Clearly, for an arbitrary metric $\bar{h}_{ij}$ one has that $\bar{r} \neq 0$.
- An idea to solve the constraint is to introduce a factor that compensates this.
- This idea leads naturally to the notion of conformal transformations.

Conformal rescalings:

Two metrics $h_{ij}, \bar{h}_{ij}$ are said to be conformally related if there exists a positive scalar $\vartheta$ (the conformal factor) such that $h_{ij} = \vartheta^4 \bar{h}_{ij}$.

The metric $\bar{h}$ will be called the background metric.

Loosely speaking, the conformal factor absorbs the overall scale of the metric.

At the level presented here, the conformal transformation introduced above is just a mathematical trick to solve equations. At a deeper level, the conformal transformation defines an equivalence class of manifolds and metrics.
A closer look at the constraint equations

Conformal rescalings and the Yamabe problem

A strategy:

- Clearly, for an arbitrary metric \( \bar{h}_{ij} \) one has that \( \bar{r} \neq 0 \).
- An idea to solve the constraint is to introduce a factor that compensates this.
- This idea leads naturally to the notion of conformal transformations.

Conformal rescalings:

- Two metrics \( h_{ij} \), \( \bar{h}_{ij} \) are said to be **conformally related** if there exists a positive scalar \( \vartheta \) (the **conformal factor**) such that

\[
    h_{ij} = \vartheta^4 \bar{h}_{ij}.
\]

- The metric \( \bar{h} \) will be called the **background** metric.
- Loosely speaking, the conformal factor absorbs the overall scale of the metric.
- At the level presented here, the conformal transformation introduced above is just a mathematical trick to solve equations. At a deeper level, the conformal transformation defines an equivalence class of manifolds and metrics.
More on conformal transformations

Transformation laws for derived objects:

- The 3-dimensional Christoffel symbols are given by
  \[
  \Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} h^{\alpha\delta} (\partial_{\beta} h_{\gamma\delta} + \partial_{\gamma} h_{\beta\delta} - \partial_{\delta} h_{\beta\gamma}).
  \]

- It follows that
  \[
  \Gamma^{\alpha}_{\beta\gamma} = \bar{\Gamma}^{\alpha}_{\beta\gamma} + 2(\delta^{\alpha}_{\beta} \partial_{\gamma} \ln \vartheta + \delta^{\alpha}_{\gamma} \partial_{\beta} \ln \vartheta - h_{\beta\gamma} h^{\alpha\delta} \partial_{\delta} \ln \vartheta).
  \]

- A lengthier computation yields the following transformation law for the Ricci tensor:
  \[
  r_{ij} = \bar{r}_{ij} - 2(\bar{D}_{i} \bar{D}_{j} \ln \vartheta + \bar{h}_{ij} \bar{h}^{lm} \bar{D}_{l} \bar{D}_{m} \ln \vartheta)
  + 4(\bar{D}_{i} \ln \vartheta \bar{D}_{j} \ln \vartheta - \bar{h}_{ij} \bar{h}^{lm} \bar{D}_{l} \ln \vartheta \bar{D}_{m} \ln \vartheta).
  \]

- Furthermore (and more importantly for our purposes) one has that
  \[
  r = \vartheta^{-4} \bar{r} - 8\bar{\vartheta}^{-5} \bar{D}_{k} \bar{D}^{k} \vartheta.
  \]

In the above expressions, \( \bar{D} \) denotes the covariant derivative of the background metric \( \bar{h}_{ij} \).
The Hamiltonian constraint and conformal rescalings:

- Using $r = 0$ in the transformation law for the Ricci scalar given above, one readily finds that

$$\bar{D}_k \bar{D}^k \vartheta - \frac{1}{8} \bar{r} \vartheta = 0.$$  

- This equation is sometimes called the **Yamabe equation** in Differential Geometry.

- Given a fixed background metric $\bar{h}_{ij}$, then it can be read as an equation for the conformal factor $\vartheta$.

- Given a solution $\vartheta$, one has that by construction $h_{ij} = \vartheta^4 \bar{h}_{ij}$ is such that $r = 0$ and one has constructed a solution to the time symmetric Einstein constraints.
The Yamabe equation is elliptic:
- the operator $\bar{D}_k \bar{D}^k$ is the Laplacian operator associated to the metric $\bar{h}_{ij}$;
- if $\bar{h}_{\alpha\beta} = \delta_{\alpha\beta}$ the flat metric in Cartesian coordinates, then
  \[
  \bar{D}_k \bar{D}^k = \delta^{\alpha\beta} \partial_\alpha \partial_\beta = \partial^2_x + \partial^2_y + \partial^2_z.
  \]

Given a linear second order elliptic equation appropriate boundary conditions ensure the existence of a unique solution on $S$. 
A closer look at the constraint equations

A further simplifying assumption: conformal flatness

What is conformal flatness?

- Choose the flat metric as background metric. That is, let

  \[ \bar{h}_{\alpha\beta} = \delta_{\alpha\beta}. \]

- In this case, the metric \( h_{\alpha\beta} = \vartheta^4 \delta_{\alpha\beta} \) is said to be **conformally flat**.

- Conformal flatness is an interesting property that Riemannian manifolds can possess. An important result is that conformal flatness is characterised locally by the vanishing of the **Cotton-York** tensor

  \[ b_{ijk} \equiv D[j r_k]_i - \frac{1}{4} h_{i[j} D_{k]} r. \]

- For example, any spherically symmetric metric can be shown to be conformally flat.

- Conformal flatness simplifies the calculations that need to be carried out.

- One has that \( \bar{r} = 0 \) so that the Yamabe equation reduces to the **flat Laplace equation**

  \[ \bar{D}_k \bar{D}^k \vartheta = 0. \]
Describing isolated systems:

- In the discussion of isolated systems (i.e. astrophysical sources) one is interested in solutions which are **asymptotically flat**. That is,

\[ \vartheta = 1 + O(r^{-1}), \quad \text{for} \quad r \to \infty, \]

where \( r^2 = x^2 + y^2 + z^2 \) is the standard radial coordinate.

- Solutions to the Laplace equation with the above asymptotic behaviour are well known. In particular, a **spherically symmetric** solution is given by

\[ \vartheta = 1 + \frac{m}{2r}, \]

where \( m \) is a constant.
Interpreting the solution:

- **Given**

  \[ \vartheta = 1 + \frac{m}{2r}, \]

  the associated solution to the Hamiltonian constraint is the 3-metric of the Schwarzschild spacetime in isotropic coordinates:

  \[ h = \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right). \]

  The above 3-metric is singular at \( r = 0 \). This singularity, is a coordinate singularity. By considering the coordinate inversion

  \[ r = \frac{m^2}{4} \frac{1}{\bar{r}}, \]

  the metric transforms into

  \[ h = \left(1 + \frac{m}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2 \theta d\phi^2\right). \]

  The inversion transforms the metric into itself —that is, it is a discrete isometry. In particular, one has that the point \( r = 0 \) is can be mapped to infinity. Thus, the metric is perfectly regular everywhere and \( r = 0 \) is, in fact, the infinity of an asymptotically flat region.
The Einstein-Rosen bridge:

- The hypersurface $S$ has a non-trivial topology—it corresponds to a *wormhole*.
- The radius given by $r = m/2$ corresponds to the minimum areal radius—this is called the *throat* of the black hole.
- The throat corresponds to the intersection of the black hole horizon with the hypersurface $S$. The inversion reflects points with respect to the throat.
The Einstein-Rosen bridge:

- The hypersurface $S$ has a non-trivial topology—it corresponds to a *wormhole*.
- The radius given by $r = m/2$ corresponds to the minimum areal radius—this is called the *throat* of the black hole.
- The throat corresponds to the intersection of the black hole horizon with the hypersurface $S$. The inversion reflects points with respect to the throat.

Embedding diagram of the Schwarzschild data:
Brill-Lindquist initial data:

- The construction described in the previous paragraphs can be extended to include an arbitrary number of black holes.
- This is made possible by the linearity of the flat Laplace equation.
- Indeed, the conformal factor

\[ \vartheta = 1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2}, \quad r_1 = |x^i - x^i_1|, \quad r_2 = |x^i - x^i_2|, \]

where \( x^i_1 \) and \( x^i_2 \) denote the (fixed) location of two black holes with bare masses \( m_1 \) and \( m_2 \).
- The solution is called the Brill-Lindquist solution.
- It describes a pair of black holes instantaneously at rest at a moment of time symmetry. This solution is much used as initial data to simulate the head-on collision of two black holes.
- One finds is that each throat connects to its own asymptotically flat region. The drawing of the corresponding 3-dimensional manifold gives 3 different sheets, each corresponding to a different asymptotically flat region.
Embedding diagram of the Brill-Lindquist data:
The Misner initial data:

- The flat Laplace equation can also be solved using the so-called **method of images** to obtain a solution with two holes and two asymptotic regions.
- This solution is known as **Misner data**.
- This solution has a reflection symmetry through the throats, and has only two (as opposed to three of the Brill-Lindquist solution) asymptotically flat regions.
- The solution can be also interpreted as a worm hole data by making suitable topological identifications.
The Misner initial data:

- The flat Laplace equation can also be solved using the so-called **method of images** to obtain a solution with two holes and two asymptotic regions.
- This solution is known as **Misner data**.
- This solution has a reflection symmetry through the throats, and has only two (as opposed to three of the Brill-Lindquist solution) asymptotically flat regions.
- The solution can be also interpreted as a worm hole data by making suitable topological identifications.

Embedding diagram of the Brill-Lindquist data:
Remarks:

- More complicated solutions to the constraint equations can be obtained by including a non-vanishing extrinsic curvature.
- In this way one can provide data for a rotating black hole or even a pair of rotating black holes spiralling around each other.
- The constraint equations in these cases have to be solved numerically.
Outline

1. The 3 + 1 decomposition of General Relativity
   - The 3+1 form of the spacetime metric

2. A closer look at the constraint equations

3. Time independent solutions
Motivation:

- A systematic analysis of solutions to the vacuum Einstein field equations must start by considering time independent solutions.
- These solutions are interpreted as describing the gravitational field in the exterior of isolated bodies at rest or in uniform rotation in an otherwise empty Universe.
- The simplest case of a time independent solution is given by the Minkowski metric.
- More sophisticated examples are given by the **Schwarzschild** and **Kerr spacetimes**.
  - The relevance of these two solutions is that they are thought to describe, in a suitable sense, the end state of black hole evolution.
Consider a scalar field on Minkowski spacetime satisfying the wave equation

\[(\Delta - \partial^2_t)\phi = 0,\]

where \(\Delta\) denote the flat Laplacian.

For time independent solutions —i.e. \(\partial_t \phi = 0\) — it follows that

\[\Delta \phi = 0.\]

An equation which is originally \textit{hyperbolic} becomes \textit{elliptic} under the assumption of time independence.

This is a generic feature that can be observed in other theories —like the Maxwell equations and the Einstein field equations.
Boundary conditions:

- The energy of the scalar field at some time $t$ is given by

  $$E(t) = \int_{S_t} \left((\partial_t \phi)^2 + |\nabla \phi|^2\right) d^3x.$$  

- In order to have finiteness of the energy one needs the boundary conditions

  $$\phi(t, x^i), \partial_t \phi(t, x^i) \to \infty, \quad \text{as} \quad |x| \to \infty.$$
Ellipticity properties:

- An important difference between hyperbolic equations and elliptic ones is that while in the former, properties of solutions can be localised and have finite propagation speed, for the latter the properties of solutions are global.

- For example, if \( \phi = O(1/r) \) as \( r \to \infty \) and \( \Delta \phi = 0 \), then it follows that \( \phi = 0 \).

- This follows from

\[
0 = \int_{\mathbb{R}^3} \phi \Delta \phi \, dx^3 = \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx^3,
\]

where Green’s identity has been used. It follows that \( |\Delta|^2 = 0 \) everywhere on \( \mathbb{R}^3 \) so that \( \phi \) is constant.

- Due to the decay conditions, it must necessarily vanish.

- This type of argument will be used repeatedly for the Einstein equations.

- In order to avoid the vanishing of \( \phi \) in this case, one needs to consider the inhomogeneous problem —that is, one needs to consider sources.
Mathematically speaking, time independence is imposed by requiring on the spacetime \((\mathcal{M}, g_{ab})\) the existence of timelike Killing vector \(\xi^a\) —the spacetime is then said to be stationary.

If, in addition, the Killing vector is hypersurface orthogonal —i.e. it is the gradient of some scalar function— then one says that \(\xi^\mu\) is static.

The Schwarzschild and Kerr solutions are, respectively, static and stationary.

Stationary solutions to the Einstein field equations allow for the possibility of rotating gravitational fields.
## Frobenius theorem:

- Let $n_a$ denote the unit normal of an hypersurface $S$.
- If $\xi^a n_a$, i.e. the Killing vector if orthogonal to $S$, then a calculation readily shows that

$$\xi[a \nabla b \xi_c] = 0.$$ 

The latter condition characterises **hypersurface orthogonality** — that is, a Killing vector is hypersurface orthogonal if and only if the previous equation holds.
The static vacuum equations (I)

The metric of a static spacetime:

- In a stationary spacetime, it is natural to choose adapted coordinates such that \( \xi^\mu \partial_\mu = \partial_t \) —that is, the time coordinate is adapted to the flow lines of the Killing vector.

- Using the Killing vector condition \( \mathcal{L}_\xi g_{ab} = 0 \) and the definitions of \( h_{ij} \) and \( K_{ij} \) one can show that

  \[
  \partial_t h_{ij} = \partial_t K_{ij} = 0.
  \]

- If the Killing vector is hypersurface orthogonal then it follows that the Killing vector has to be proportional to the normal to the hypersurface \( S \):

  \[
  \xi_\mu = \alpha \nabla_\mu t
  \]

- However, the Killing vector can be decomposed in a lapse and a shift part:

  \[
  \xi^a = N n^a + \beta^a.
  \]

- Comparing both expressions one necessarily has that \( \beta^\alpha = 0 \).

- Thus, one has that

  \[
  g = -\alpha^2 dt^2 + h_{\alpha\beta} dx^\alpha dx^\beta,
  \]

  with \( h_{\alpha\beta} \) time independent.
The static vacuum equations (II)

Vanishing of the extrinsic curvature:

- The time evolution equation for $h_{ij}$ then takes the form

$$\partial_t h_{ij} = -2\alpha K_{ij} = 0.$$ 

- As the lapse cannot vanish one has that

$$K_{ij} = 0.$$ 

That is, the hypersurfaces of the foliation adapted to the static Killing vector have no extrinsic curvature —this property is preserved as $\partial_t K_{ij} = 0$. 

Vacuum static solutions are characterised solely in terms of the lapse $\alpha$ and the 3-metric $h_{ij}$.

In order to obtain equations for these quantities one considers the Hamiltonian constraint and the evolution equation for $K_{ij}$.

Setting $K_{ij} = \partial_t K_{ij} = 0$ readily yields

$$D_i D_j \alpha = r_{ij},$$

$$r = 0,$$

where, as before, $r$ denotes the Ricci tensor of the 3-metric $h_{ij}$. These equations are known as the static vacuum Einstein equations.
Assumptions and boundary conditions:

- As a first example of the content and implications of the static equations let $\mathcal{S} \approx \mathbb{R}^3$
  - i.e. the hypersurface has the topology of Euclidean space.
- Suppose that the fields $\alpha$ and $h_{\alpha\beta}$ decay at infinity in such a way that
  $$\alpha \to 1, \quad h_{\alpha\beta} - \delta_{\alpha\beta} \to 0, \quad \text{as} \quad r \to \infty.$$  

- The first condition essentially means that it is assumed that the Killing vector behaves asymptotically like the static Killing vector of Minkowski spacetime.
- The second condition means that the 3-metric is assumed to be asymptotically flat (Euclidean) at infinity.
Ellipticity and the static equations:

- Taking traces of the first static equation and using the second equation it follows that
  \[ \Delta \alpha = D_k D^k \alpha = 0. \]

- Now, consider
  \[ 0 = \int_S \alpha \Delta \alpha d^3x = \int_S |D \alpha|^2 d^3x, \]
  again, as a consequence of Green’s identity.

- Thus
  \[ |D \alpha|^2 = h^{ij} D_i \alpha D_j \alpha = 0, \]
  from where it follows that \( \alpha \) is a constant.

- Using the asymptotic condition \( \alpha \to 1 \) it follows \( \alpha = 1 \) everywhere.
Flatness of the 3-metric:

- Using the first static equation one concludes that
  \[ r_{ij} = 0. \]

- In 3-dimensions the Ricci tensor determines fully the curvature of the manifold. Thus
  \[ r_{ijkl} = 0. \]

  That is, \( h_{\alpha\beta} = \delta_{\alpha\beta} \) —the Euclidean flat metric.

- The solution we have obtained then is
  \[ g = -dt^2 + \delta_{\alpha\beta} dx^\alpha dx^\beta. \]

This solution is the Minkowski spacetime! This result is known as **Licnerowicz’s theorem**.
The only globally regular static solution to the Einstein equations with $S$ having trivial topology (i.e. $S \approx \mathbb{R}^3$) and such that

$$\alpha \to 1, \quad h_{\alpha\beta} - \delta_{\alpha\beta} \to 0, \quad \text{as} \quad r \to \infty$$

is the Minkowski spacetime.
Properties of the static equations: Licnerowicz theorem (IV)

Theorem

The only globally regular static solution to the Einstein equations with $S$ having trivial topology (i.e. $S \approx \mathbb{R}^3$) and such that

$$\alpha \to 1, \quad h_{\alpha\beta} - \delta_{\alpha\beta} \to 0, \quad \text{as} \quad r \to \infty$$

is the Minkowski spacetime.

Morally:

- The above theorem demonstrates the rigidity of the Einstein field equations.
- In order to obtain more interesting regular solutions, one requires either some matter sources or a non-trivial topology for $S$ as in the case of the Schwarzschild spacetime — recall the Einstein-Rosen bridge!
- The result can be interpreted as a first, very basic uniqueness black hole result.
- If one wants to have a black hole solution one needs non-trivial topology!
Further properties of static solutions: leading asymptotic behaviour

Question:

- An important question when analysing static spacetimes is to analyse their asymptotic behaviour beyond the prescribed boundary conditions.
- Can one say more?

Theorem (Beig, 1980)

Every static vacuum solution to the Einstein equations satisfying

\[ \alpha \rightarrow 1, \quad h_{\alpha\beta} - \delta_{\alpha\beta} \rightarrow 0, \quad \text{as} \quad r \rightarrow \infty \]

is Schwarzschildean to leading order in \( \frac{1}{r} \). That is,

\[ \alpha^2 = 1 - \frac{2m}{r} + O\left(\frac{1}{r^2}\right) \]

\[ h_{\alpha\beta} - \delta_{\alpha\beta} = \frac{2m}{r} \delta_{\alpha\beta} + O\left(\frac{1}{r^2}\right) \]

Remarks:

Notice that in the previous result the regularity of \( S \) is not required. Also, there could be bounded sources somewhere in the interior.
Further properties of static solutions: leading asymptotic behaviour

Question:

- An important question when analysing static spacetimes is to analyse their asymptotic behaviour beyond the prescribed boundary conditions.
- Can one say more?

Theorem (Beig, 1980)

*Every static vacuum solution to the Einstein equations satisfying*

\[ \alpha \to 1, \quad h_{\alpha \beta} - \delta_{\alpha \beta} \to 0, \quad \text{as} \quad r \to \infty \]

*is Schwarzschildian to leading order in $1/r$. That is,*

\[ \alpha^2 = 1 - \frac{2m}{r} + O(1/r^2), \quad h_{\alpha \beta} - \delta_{\alpha \beta} = \frac{2m}{r} \delta_{\alpha \beta} + O(1/r^2). \]

Remarks:

- Notice that in the previous result the regularity of $S$ is not required. Also, there could be bounded sources somewhere in the interior.
Further properties of static solutions: multipole moments

Defining multipole moments:

- The lapse $\alpha$ can be interpreted as relativistic generalisation of a Newtonian potential.
- The previous theorem on the leading behaviour of static solutions theorem can be improved to include higher order multipoles.
- These lead to a multipolar expansion of the gravitational field.
- These multipoles characterise in a unique manner static solutions.
Defining multipole moments:

- The lapse $\alpha$ can be interpreted as relativistic generalisation of a Newtonian potential.
- The previous theorem on the leading behaviour of static solutions theorem can be improved to include higher order multipoles.
- These lead to a multipolar expansion of the gravitational field.
- These multipoles characterise in a unique manner static solutions.

Theorem (Beig & Simon, 1981; Friedrich 2006)

Given an asymptotically flat static solution to the Einstein vacuum equations, one obtains a unique sequence of multipole moments. Conversely, given a sequence of multipole moments, if the lapse constructed from this sequence exists, there exists a unique static spacetime associated to these multipoles.