Mathematical problems of General Relativity
Lecture 2

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LTCC Course LMS
The $3+1$ decomposition of General Relativity

1. Submanifolds of spacetime
2. Foliations of spacetime
3. The intrinsic metric of an hypersurface
4. The extrinsic curvature of an hypersurface
5. The Gauss-Codazzi and Codazzi-Mainardi equations
6. The constraint equations of General Relativity
7. The ADM-evolution equations
The 3 + 1 decomposition of General Relativity

Submanifolds of spacetime

Foliations of spacetime

The intrinsic metric of an hypersurface

The extrinsic curvature of an hypersurface

The Gauss-Codazzi and Codazzi-Mainardi equations

The constraint equations of General Relativity

The ADM-evolution equations
Submanifolds

Intuitive definition:

- A **submanifold** of $\mathcal{M}$, is a set $\mathcal{N} \subset \mathcal{M}$ which inherits a manifold structure from $\mathcal{M}$.
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Embeddings:

- An embedding map $\varphi : \mathcal{N} \to \mathcal{M}$ which is injective and structure preserving;
- The restriction $\varphi : \mathcal{N} \to \varphi(\mathcal{N})$ is a diffeomorphism.
Submanifolds

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Rigorous definition of submanifold:

- In terms of the above concepts, a submanifold \( \mathcal{N} \) is the image \( \varphi(\mathcal{N}) \subset \mathcal{M} \) of a \( k \)-dimensional manifold \( (k < n) \).
- Very often it is convenient to identify \( \mathcal{N} \) with \( \varphi(\mathcal{N}) \).
- In what follows we will mostly be concerned with 3-dimensional submanifolds. It is customary to call these **hypersurfaces**.
The $3+1$ decomposition of General Relativity

Outline

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Foliations

Globally hyperbolic spacetimes:

- In what follows, we assume that the spacetime $(\mathcal{M}, g_{ab})$ is **globally hyperbolic**.
- That is, we assume that its topology is that of $\mathbb{R} \times S$, where $S$ is an orientable 3-dimensional manifold.
- Globally hyperbolic spacetimes are the natural class of spacetimes on which to formulate a Cauchy problem.

Definition of a foliation:

- A spacetime is said to be **foliated** by (non-intersecting) hypersurfaces $S_t$, $t \in \mathbb{R}$ if
  \[ \mathcal{M} = \bigcup_{t \in \mathbb{R}} S_t, \]
  where we identify the leaves $S_t$ with $\{t\} \times S$.
- It is customary to think of the hypersurface $S_0$ as an initial hypersurface on which the initial information giving rise to the spacetime is to be prescribed.
Time functions

Definition:

- In what follows it will be convenient to assume that the hypersurfaces $S_t$ arise as the level surfaces of a scalar function $t$ which will be interpreted as a **global time function**.
- From $t$ one can define the covector

$$\omega_a = \nabla_a t.$$ 

By construction $\omega_a$ denotes the normal to the leaves $S_t$ of the foliation.
- The covector $\omega_a$ is **closed** —that is,

$$\nabla[a \omega_b] = \nabla[a \nabla_b] t = 0.$$
The lapse function

Definition:

- From $\omega^a$ one defines a scalar $\alpha$ called the **lapse function** via
  \[ g^{ab} \nabla_a t \nabla_b t = \nabla^a t \nabla_a t \equiv -1/\alpha^2. \]
- The lapse measures how much proper time elapses between neighbouring time slices along the direction given by the normal vector $\omega^a \equiv g^{ab} \omega_b$.
- Assume that $\alpha > 0$ so that $\omega^a$. Notice that $\omega^a$ is assumed to be timelike so that the hypersurfaces $S_t$ are spacelike.
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Unit normal:
- In what follows we define the **unit normal** $n_a$ via

$$n_a \equiv -\alpha \omega_a.$$ 

- The minus sign in the last definition is chosen so that $n^a$ points in the direction of increasing $t$.
- One can readily verify that $n^a n_a = -1$.
- One thinks of $n^a$ as the 4-velocity of a normal observer whose worldline is always orthogonal to the hypersurfaces $S_t$. 
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The intrinsic metric (I)

Definition:

- The spacetime metric $g_{ab}$ induces a 3-dimensional Riemannian metric $h_{ij}$ on $S_t$.
- The relation between $g_{ab}$ and $h_{ab}$ is given by
  \[ h_{ab} = g_{ab} + n^a n^b. \]
- In the previous formula we regard the 3-metric as an object living on spacetime.

Properties:

- The tensor $h_{ab}$ is purely spatial — i.e. it has no component along $n^a$.
- Contracting with the normal:
  \[ n^a h_{ab} = n^a g_{ab} + n_a n^a n_b = n_b - n_b = 0, \]
- The inverse 3-metric $h^{ab}$ is obtained by raising indices with
  \[ h^{ab} = g^{ab} + n^a n^b. \]
Use as a projector:

- The 3-metric $h_{ab}$ can be used to project all geometric objects along the direction given by $n^a$.
- Effectively, $h_{ab}$ decomposes tensors into a **purely spatial part** which lies on the hypersurfaces $S_t$ and a **timelike part** normal to the hypersurface.
- In actual computations it is convenient to consider

$$h_a^b = \delta_a^b + n_a n^b.$$ 

- Given a tensor $T_{ab}$ its spatial part, to be denoted by $T^\perp_{ab}$ is defined to be

$$T^\perp_{ab} \equiv h_a^c h_b^d T_{cd}.$$
The normal projector

Definition:

- One can also define a *normal projector* $N_{ab}^b$ as
  \[ N_{ab}^b \equiv -n_a n^b = \delta_{ab} - h_{ab}. \]

- In terms of these operators an arbitrary projector can be decomposed as
  \[ v^a = \delta^a b v^b = (h_{ab} + N_{ab}) = v^a - n^a n_b v^b. \]
Covariant derivatives on hypersurfaces (I)

A definition of a covariant derivative:

- The 3-metric $h_{ij}$ defines in a unique manner a covariant derivative $D_i$ — the Levi-Civita connection of $h_{ij}$.
- Work from a 4-dimensional (spacetime) perspective so that we write $D_a$.
- One requires $D_a$ to be torsion-free and compatible with the metric $h_{ab}$.
- For a scalar $\phi$
  \[ D_a \phi \equiv h^b \nabla_b \phi, \]
  and, say, for a $(1, 1)$ tensor
  \[ D_a T^b_c \equiv h_a^d h^b_e h^f_c \nabla_d T^e_f, \]
  with an obvious extension to other tensors.
- In coordinates, the covariant derivative $D_a$ is associated to the spatial Christoffel symbols
  \[ \gamma^\mu_{\nu\lambda} = \frac{1}{2} h^{\mu\rho} (\partial_\nu h_{\rho\lambda} + \partial_\lambda h_{\nu\rho} - \partial_\rho h_{\nu\lambda}). \]
The curvature of $D_a$:

- Being a covariant derivative, one can naturally associate a curvature tensor $r^a_{bcd}$ to $D_a$ by considering its commutator:

$$D_a D_b v^c - D_b D_a v^c = r^c_{dab} v^d$$

One can verify that $r^c_{dab} n^d = 0$.

- Similarly, one can define the Ricci tensors and scalar as

$$r_{db} \equiv r^c_{dcb}, \quad r \equiv g^{ab} r_{ab}.$$
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The extrinsic curvature (I)

Motivation:

- The Einstein field equation $R_{ab} = 0$ imposes some conditions on the 4-dimensional Riemann tensor $R^{a}_{bcd}$.
- In order to understand the implications of the Einstein equations on an hypersurface one needs to decompose $R^{a}_{bcd}$ into spatial parts. This decomposition naturally involves $r^{a}_{bcd}$.
- The tensor $r^{a}_{bcd}$ measures the intrinsic curvature of the hypersurface $S_t$. This tensor provides no information about how $S_t$ fits in $(\mathcal{M}, g_{ab})$.
- The missing information is contained in the so-called extrinsic curvature.
The extrinsic curvature (II)

Definition:

- The extrinsic curvature is defined as the following projection of the spacetime covariant derivative of the normal to $S_t$:

$$K_{ab} \equiv -h_a^c h_b^d \nabla_{(c} n_{d)} = -h_a^c h_b^d \nabla_{c} n_{d}.$$

The second equality follows from the fact that $n_a$ is rotation free.

- By construction the extrinsic curvature is symmetric and purely spatial.

- It measures how the normal to the hypersurface changes from point to point.

- It also measures the rate at which the hypersurface deforms as it is carried along the normal —Ricci identity.
The acceleration

Definition:

- The *acceleration* of a foliation is defined via

\[ a_a \equiv n^b \nabla_b n_a. \]

- Using \( n^d \nabla_c \nabla_d = 0 \), one can compute

\[
K_{ab} = -h_a^c h_b^d \nabla_c n_d \\
= - (\delta_a^c + n_a n^c)(\delta_b^d + n_b n^d) \\
= - (\delta_a^c + n_a n^c)\delta_b^d \nabla_c n_d \\
= - \nabla_a n_b - n_a a_b.
\]
The Lie derivative of the intrinsic metric:

- One computes

\[ \mathcal{L}_n h_{ab} = \mathcal{L}_n (g_{ab} + n_a n_b) \]
\[ = 2\nabla_a n_b + n_a \mathcal{L}_n n_b + n_b \mathcal{L}_n n_a \]
\[ = 2(\nabla_a n_b + n_a n_b) \]
\[ = -2K_{ab}. \]
Mean curvature

Definition:

- A related object is the so-called **mean curvature**:
  \[ K \equiv g^{ab} K_{ab} = h^{ab} K_{ab}. \]

- One can compute (exercise):
  \[ K = -\mathcal{L}_n (\ln \det h). \]

- Thus the mean curvature measures the fractional change in 3-dimensional volume along the normal \( n^a \).

- An hypersurface for which \( K = 0 \) everywhere is called **maximal** —it encloses maximum volume for a given area.
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The Gauss-Codazzi equation

Motivation:

- Given the extrinsic curvature of an hypersurface $S_t$, we now look how this relates to the curvature of spacetime.
- A computation using the definitions of the previous section shows that

$$D_a D_b v^c = h_a^p h_b^q h_r^c \nabla_p \nabla_q v^r - K_{ab} h_r^c n^p \nabla_p v^r - K_a^c K_{bp} v^p.$$  

- Combining with the commutator

$$D_a D_b v^c - D_b D_a v^c = r^c_{dab} v^d,$$

after some manipulations one obtains

$$r_{abcd} + K_{ac} K_{bd} - K_{ad} K_{cb} = h_a^p h_b^q h_c^r h_d^s R_{pqrs}.$$  

- This equation is called the **Gauss-Codazzi equation**. It relates the spatial projection of the spacetime curvature tensor to the 3-dimensional curvature.
The Codazzi-Mainardi equation

Motivation:

- A further important identity arises from considering projections of $R_{abcd}$ along the normal direction. This involves a spatial derivative of the extrinsic curvature.

- One has that

  $$D_a K_{bc} = h_a^p h_b^q h_c^r \nabla_p K_{qr}.$$ 

- From this expression after some manipulations one can deduce

  $$D_b K_{ac} - D_a K_{bc} = h_a^p h_b^q h_c^r n^s R_{pqrs}.$$ 

- This equation is called the **Codazzi-Mainardi equation**.
The Codazzi-Mainardi equation

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In the sequel:

- In the sequel, we explore the consequences of the Gauss-Codazzi and Codazzi-Mainardi equations for the initial value problem in General Relativity.
- These give rise to the so-called **constraint equations of General Relativity**.
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Strategy:

- The $3+1$ decomposition of the Einstein field equations allows to identify the intrinsic metric and the extrinsic curvature of an initial hypersurface $S_0$ as the initial data to be prescribed for the evolution equations of General Relativity.

- In what follows we will make use of the Gauss-Codazzi and the Codazzi-Mainardi equations to extract the consequences of the vacuum Einstein field equations

$$R_{ab} = 0$$

on a hypersurface $S$. 
The Hamiltonian constraint (I)

Derivation of the equation:

- Contracting the Gauss-Codazzi equation one finds that

\[
h^{pr} h_b^q h_d^s R_{pqrs} = r_{bd} + KK_{bd} - K^c_d K_{cb},
\]

where \( K \equiv h^{ab} K_{ab} \) denotes the trace of the extrinsic curvature.

- A further contraction then yields

\[
h^{pr} h^{qs} R_{pqrs} = r + K^2 - K_{ab} K^{ab}.
\]

- Now, the left-hand side can be expanded into

\[
h^{pr} h^{qs} R_{pqrs} = (g^{pr} + n^p n^s)(g^{qs} + n^q n^s)
\]

\[
= R + 2n^p n^r R_{pr} + n^p n^r n^q n^s R_{pqrs} = 0.
\]

The last term vanishes because of the symmetries of the Riemann tensor.
The Hamiltonian constraint (II)

Summarising

Combining the equations from the previous calculations one obtains the so-called **Hamiltonian constraint**:

\[ r + K^2 - K_{ab}K^{ab} = 0. \]
The momentum constraint

Derivation:

- Contracting once the Codazzi-Mainardi equation one has that

\[ D^b K_{ab} - D_a K = h_ah^p q^r n^s R_{pqrs}. \]

- The right hand side of this equation can be, in turn, expanded as

\[
\begin{align*}
  h_ah^p q^r n^s R_{pqrs} &= -h_ah^p (g^{qr} + n^p n^r)n^s R_{qprs} \\
  &= -h_ah^p n^s R_{ps} - h_ah^p n^q n^r n^s R_{pqrs} = 0,
\end{align*}
\]

where in the last equality one makes use, again, of the vacuum Equations and the symmetries of the Riemann tensor.
The momentum constraint

**Derivation:**

- Contracting once the Codazzi-Mainardi equation one has that
  \[ D^b K_{ab} - D_a K = h_a^p h^{qr} n^s R_{pqrs}. \]
- The right hand side of this equation can be, in turn, expanded as
  \[ h_a^p h^{qr} n^s R_{pqrs} = -h_a^p (g^{qr} + n^p n^r) n^s R_{qprs} \]
  \[ = -h_a^p n^s R_{ps} - h_a^p n^q n^r n^s R_{pqrs} = 0, \]
  where in the last equality one makes use, again, of the vacuum Equations and the symmetries of the Riemann tensor.

**Summarising:**

Combining the previous expressions one obtains the so-called **momentum constraint**:

\[ D^b K_{ab} - D_a K = 0. \]
Initial data and the constraint equations

Discussion:

- The Hamiltonian and momentum constraint involve only the 3-dimensional intrinsic metric, the extrinsic curvature and their spatial derivatives.
- They are the conditions that allow a 3-dimensional slice with data \((h_{ab}, K_{ab})\) to be embedded in a 4-dimensional spacetime \((\mathcal{M}, g_{ab})\).
- The existence of the constraint equations implies that the data for the Einstein field equations cannot be prescribed freely.
Initial data and the constraint equations

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- The Hamiltonian and momentum constraint involve only the 3-dimensional intrinsic metric, the extrinsic curvature and their spatial derivatives.
- They are the conditions that allow a 3-dimensional slice with data $(h_{ab}, K_{ab})$ to be embedded in a 4-dimensional spacetime $(\mathcal{M}, g_{ab})$.
- The existence of the constraint equations implies that the data for the Einstein field equations cannot be prescribed freely.

Remark:
An important point still to be clarified is the sense in which the fields $h_{ab}$ and $K_{ab}$ correspond to data for the Einstein field equations. To see this, one has to analyse the evolution equations implied by the Einstein field equations.
The constraint equations for the electromagnetic field (I)

A source of insight:

- The equations of other physical theories also imply constraint equations. The classical example in this respect is given by the Maxwell equations.

- In order to analyse the constraint equations implied by the Maxwell equations it is convenient to introduce the electric and magnetic parts of the Faraday tensor $F_{ab}$:

\[
E_a \equiv F_{ab} n^b, \quad B_a \equiv \frac{1}{2} \epsilon_{abcd} F_{cd} n^b = F^*_{ab} n^b.
\]

- A calculation then shows that the Maxwell equations imply the constraint equations

\[
D^a E_a = 0, \quad D^a B_a = 0.
\]

These constraints correspond to the well-known Gauss laws for the electric and magnetic fields.
Summarising:

- Thus, it follows that data for the Maxwell equations **cannot be prescribed freely**. The initial value of the electric and magnetic parts of the Faraday tensor must be divergence free.

- Notice, by contrast that the wave equation for a scalar field $\phi$ implies no constraint equations. Thus, the data for this equation **can be prescribed freely**.
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The Ricci equation

Strategy:

- In a previous lecture we have seen that the Einstein equations imply a wave equation for the components of the metric tensor. These equations are second order.

- In order to obtain evolution equations which are of first order one needs a geometric identity relating the Lie derivative of the extrinsic curvature in the direction to the normal of the foliation.

Derivation:
Starting from \( L_n K_{ab} = n^c \nabla c K_{ab} + 2K_c^{(a} \nabla^{b)} n^c \), some manipulations (see the notes) lead to the so-called Ricci equation:

\[
L_a K_{ab} = n^d n^c h_{a}^q h_{b}^r R_{drcq} - \frac{1}{\alpha} D_a D_b \alpha - K_b^{c} K_c^{a}.
\]

Geometrically, this equation relates the derivative of the extrinsic curvature in the normal direction to an hypersurface \( S \) to a time projection of the Riemann tensor.
The Ricci equation

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- In order to obtain evolution equations which are of **first order** one needs a geometric identity relating the Lie derivative of the extrinsic curvature in the direction to the normal of the foliation.

Derivation:

- Starting from

\[ \mathcal{L}_n K_{ab} = n^c \nabla_c K_{ab} + 2 K_{c(a} \nabla_{b)} n^c, \]

some manipulations (see the notes) lead the so-called **Ricci equation**:

\[ \mathcal{L}_a K_{ab} = n^d n^c h_a{}^q h_b{}^r R_{drcq} - \frac{1}{\alpha} D_a D_b \alpha - K_{b}{}^{c} K_{ac}. \]

Geometrically, this equation relates the derivative of the extrinsic curvature in the normal direction to an hypersurface \( S \) to a time projection of the Riemann tensor.
The time vector:

- The discussion from the previous paragraphs suggests that the Einstein field equations will imply an **evolution** of the data \((h_{ab}, K_{ab})\).
- Assumed that the spacetime \((\mathcal{M}, g_{ab})\) is foliated by a time function \(t\) whose level surfaces corresponds to the leaves of the foliation.
- Recalling that \(\omega_a = \nabla_a t\), we consider now a vector \(t^a\) (the **time vector**) such that

\[
t^a = \alpha n^a + \beta^a, \quad \beta_a n^a = 0.
\]
The shift vector:

- The vector $\beta^a$ is called the \textbf{shift vector}.
- The time vector $t^a$ will be used to \textbf{propagate coordinates} from one time slice to another.
- In other words, $t^a$ connects points with the same spatial coordinate — hence, the shift vector measures the amount by which the spatial coordinates are shifted within a slice with respect to the normal vector.
The time vector and the shift vector (III)

Gauge functions:

- Together, the lapse and shift determine how coordinates evolve in time. The choice of these functions is fairly arbitrary and hence they are known as **gauge functions**.
- The lapse function reflects the freedom to choose the sequence of time slices, pushing them forward by different amounts of proper time at different spatial points on a slice—this idea is usually known as the **many-fingered nature of time**.
- The shift vector reflects the freedom to relabel spatial coordinates on each slice in an arbitrary way.
- Observers **at rest** relative to the slices follow the normal congruence $n^a$ and are called **Eulerian observers**, while observers following the congruence $t^a$ are called **coordinate observers**.
- It is observed that as a consequence of the previous definitions one has that $t^a \nabla_a t = 1$ so that the integral curves of $t^a$ are naturally parametrised by $t$. 
The evolution equation for the 3-metric

Derivation of the equation:

- Recalling that
  \[ K_{ab} = -\frac{1}{2} \mathcal{L}_n h_{ab} \]
  and using the equation \( t^a = \alpha n^a + \beta^a \) one concludes that
  \[ \mathcal{L}_t h_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta h_{ab}, \]
  where it has been used that
  \[ \mathcal{L}_t h_{ab} = \mathcal{L}_{\alpha n + \beta} h_{ab} = \alpha \mathcal{L}_n h_{ab} + \mathcal{L}_\beta h_{ab}. \]

- This equation will be interpreted as an evolution equation for the intrinsic metric \( h_{ab} \).
Evolution equation for the second fundamental form

Derivation of the equation:

- In order to construct a similar equation for the extrinsic curvature one makes use of the Ricci equation.

- It is noticed that

\[ n^d n^c h^q_a h^r_b R_{drcq} = h^{cd} h^q_a h^r_b R_{drcq} - h^q_a h^r_b R_{rq} = h^{cd} h^q_a h^r_b R_{drcq}, \]

where to obtain the second equality \( R_{ab} = 0 \) has been used. The remaining term, \( h^{cd} h^q_a h^r_b R_{drcq} \) is dealt with using the Gauss-Codazzi equation.

- Finally, noticing that

\[ \mathcal{L}_t K_{ab} = \mathcal{L}_\alpha n + \beta K_{ab} = \alpha \mathcal{L}_n K_{ab} + \mathcal{L}_\beta K_{ab}, \]

one concludes that

\[ \mathcal{L}_t K_{ab} = -D_a D_b \alpha + \alpha (r_{ab} - 2K_{ac} K^c_b + K K_{ab}) + \mathcal{L}_\beta K_{ab}. \]

This is the desired evolution equation for \( K_{ab} \).
The evolution equations deduced in the previous slices determine the evolution of the data \((h_{ab}, K_{ab})\). These equations are usually known as the ADM (Arnowitz-Deser-Misner) equations.

Together with the constraint equations they are completely equivalent to the vacuum Einstein field equations.

The ADM evolution equations are first order equations —contrast with the wave equation for the components of the metric \(g_{ab}\) discussed in a previous lecture. However, the equations are not hyperbolic!

Thus, one cannot apply directly the standard PDE theory to assert existence of solutions. Nevertheless, there are some more complicated versions which do have the hyperbolicity property.
A source of insight:

- As in the case of the constraint equations, it is useful to compare with the Maxwell field equations.
- Making use of the electric and magnetic part of the Faraday tensor, a computation of $\mathcal{L}_t E_a$ and $\mathcal{L}_t B_a$ together with the Maxwell equations allows to show that

$$\mathcal{L}_t E_a = \epsilon_{abc} D^b E^c + \mathcal{L}_\beta E_a,$$
$$\mathcal{L}_t B_a = -\epsilon_{abc} D^b B^c + \mathcal{L}_\beta B_a.$$ 

- Notice the similarity with the ADM equations!