Mathematical problems of General Relativity

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LTCC Course LMS

Outline

Objectives and content

Objectives:

- Provide a discussion of General Relativity as an initial value problem.
- Provide an introduction to applied methods of Differential Geometry and partial differential equations.
- Give an overview of main ideas and methods of mathematical General Relativity;

Topics to be covered

1. A review of Differential Geometry
2. A survey of General Relativity
3. The Einstein equation as a wave equation
4. The $3+1$ decomposition of General Relativity
5. The constraint equations of General Relativity
6. The ADM evolution equations
7. Time independent solutions
8. Energy and momentum in General Relativity (if time permits)

Resources and further material

Notes:

- Available at: www.maths.qmul.ac.uk/∼jav/LTCC
- These include notes of the lectures, slides and an extended overview of Differential Geometry —all comments abut these welcome!

Problems:

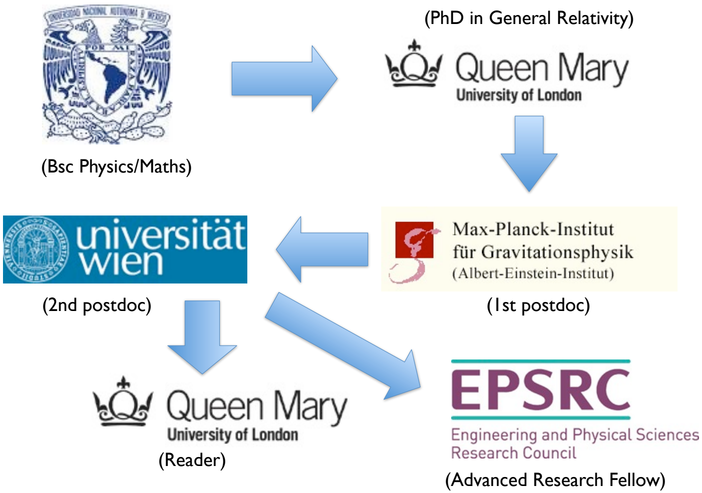
- 4 problem sheets will be provided.
- Mainly to elaborate one calculations briefly discussed in the lectures.

Assessment

The course contains a *light assessment* consisting of a problem sheet to take home and to be handed back two weeks after.

About me

About me:



Outline

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Manifolds (I)

Definition:

- The basic concept in Differential Geometry is that of a **differentiable manifold** (or manifold for short).
- A manifold $\mathcal{M}$ is essentially a (topological) space that can be covered by a collection of *charts* $(\mathcal{U}, \phi)$ where $\mathcal{U} \subset \mathcal{M}$ is an open subset and $\phi : \mathcal{U} \to \mathbb{R}^n$ for some n is a smooth injective (one-to-one) mapping.
- The notion of a manifold requires certain compatibility between overlapping charts.
- In what follows, for simplicity and unless otherwise stated, it is assumed that all structures are **smooth**.
- Attention will be restricted to manifolds of dimensions 4 and 3.

Manifolds (II)

Local coordinates:

Given $p \in \mathcal{U}$ one writes

$$\phi(p) = (x^1, \ldots, x^n).$$

The $(x^\mu) = (x^1, \ldots, x^n)$ are called the **local coordinates** on $\mathcal{U}$.

Orientability:

A manifold $\mathcal{M}$ is said to be **orientable** if the Jacobian of the transformation between overlapping charts is positive.

Scalar fields over a manifold:

A **scalar field** over $\mathcal{M}$ is a smooth function $f : \mathcal{M} \to \mathbb{R}$. The set of scalar fields over $\mathcal{M}$ will be denoted by $\mathfrak{X}(\mathcal{M})$.

Curves on manifold

Definition:

- A **curve** is a smooth map $\gamma : I \to \mathcal{M}$ with $I \subset \mathbb{R}$.
- In terms of coordinates (x^μ) defined over a chart of $\mathcal{M}$ one writes the curve as

$$x^\mu(\lambda) = (x^1(\lambda), \ldots, x^n(\lambda)),$$

where $\lambda \in I$ is the parameter of the curve.

Vectors on a manifold (I)

Tangent vector:

- The concept of **tangent vector** formalises the physical notion of velocity.
- In local coordinates, the tangent vector to the curve $x^\mu(\lambda)$ is given by

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}.$$

- In modern Differential Geometry one identifies vectors with homogeneous first order differential operators acting on scalar fields over $\mathcal{M}$.
- This approach allow to encode in a simple manner the *classical* transformation properties of vectors between charts.
- Following this perspective, in local coordinates a vector field will be written as

$$v^\mu \partial_\mu.$$

Vectors on a manifold (II)

Abstract index notation:

- In what follows we will mostly make use of the **abstract index notation** to denote vectors and tensors.
- A generic vector will in this formalism denoted as v^a.
- The role of the superindex in this notation is to indicate the character of the object in question.
- For the components in some coordinate system (x^μ) write v^μ.

Tangent space and tangent bundle:

- The set of vectors at a point p of $\mathcal{M}$ is the **tangent space at** p, $T_p\mathcal{M}$.
- A (smooth) prescription of a vector at every point of $\mathcal{M}$ is called a **vector field**.
- The collection of all tangent spaces on $\mathcal{M}$ is called the **tangent bundle** $T\mathcal{M}$.

Covectors

Definition:

- A **covector** (or 1-form) is real valued function of a vector.
- In abstract index notation denoted by ω_a.
- The action of ω_a on v^a will be denoted by $\omega_a v^a \in \mathfrak{X}(\mathcal{M})$.

Cotangent space

- The set of covectors at a point $p \in \mathcal{M}$ is the *cotangent space* $T^*_p\mathcal{M}$.
- The set of all cotangent spaces on $\mathcal{M}$ is the *cotangent bundle* $T^*\mathcal{M}$.

Higher rank tensors

Definition:

- Higher rank objects (tensors) can be constructed by analogy.
- A **tensor** of type (m, n) is a real-valued functions of m covectors and n vectors that are linear in all their arguments.
- For example, the tensor $T^{ab}{}_c$ is of type $(2, 1)$.
- Traditionally, superindices in a tensor are called **contravariant** while subindices ones are called **covariant**.

Symmetric and antisymmetric tensors:

- A tensor is **symmetric** if it remains unchanged under the interchange of two of its arguments $T_{ab} = T_{ba}$.
- A tensor is **antisymmetric** if it changes sign with an interchange of a pair of arguments as in $S_{abc} = -S_{acb}$.
- The **symmetric and antisymmetric parts** of a tensor can be constructed by adding together all possible permutations with the appropriate signs. For example

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}), \qquad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}).$$

Outline

Metric tensors (I)

Definition:

- A **metric** on $\mathcal{M}$ is a non-degenerate symmetric $(0,2)$ tensor field g_{ab}.
- **Non-degenerate:** if $g_{ab}u^a v^b = 0$ for all u^a if and only if $v^a = 0$.
- The metric encodes the geometric notions of orthogonality and norm of a vector.
- The norm of a vector is given by $|v|^2 = g_{ab}v^a v^b$
- If $g_{ab}v^a u^a = 0$, then v^a and u^a are said to be **orthogonal**.

Riemannian and Lorentzian metrics

- In terms of a coordinate system (x^μ) the components of g_{ab}, $g_{\mu\nu}$, are a $n \times n$ matrix. Because of symmetry, this matrix has n real eigenvalues.
- The **signature** of g_{ab} is the difference between the number of positive and negative eigenvalues.
- If the signature is $\pm n$ then one has a **Riemannian metric**.
- If the signature is $\pm(n-2)$ then the metric is said to be **Lorentzian**.

Metric tensors (II)

Index gymnastics:

- A metric g_{ab} can be used to define a one-to-one correspondence between vectors and covectors.
- In local coordinates denote by $g^{\mu\nu}$ the inverse of $g_{\mu\nu}$. This defines a $(2,0)$ tensor which we denote by g^{ab}.
- By construction $g_{ab}g^{bc} = \delta_a{}^c$ where $\delta_a{}^c$ is the **Kroneker delta**.
- Given a vector v^a one defines $v_a \equiv g_{ab}v^a$.
- Similarly, given a covector ω_a one can define $\omega^a \equiv g^{ab}\omega_b$.

Remarks for Lorentzian metrics

Classifying vectors according to their causal nature:

- In these lectures all Lorentzian metrics will be defined on a 4-dimensional manifold and will be assumed to have signature 2 —that is, one has one negative eigenvalue and 3 positive ones.
- A Lorentzian metric can be used to classify vectors according to the sign of their norm.
  - v^a is said to be **timelike** if $g_{ab}v^a v^b < 0$;
  - v^a is said to be **null** if $g_{ab}v^a v^b = 0$;
  - v^a is said to be **spacelike** if $g_{ab}v^a v^b > 0$.

The Levi-Civita connection (I)

Covariant derivatives

- A **covariant derivative** is a notion of derivative with tensorial properties.
- A metric g_{ab} allows to define a covariant derivative ∇_a over $\mathcal{M}$ —the so-called **Levi-Civita connection**.
- The covariant derivative of a vector v^a is denoted by $\nabla_a v^b$. For a covector ω_b one writes $\nabla_a \omega_b$.

The Christoffel symbols

- Explicit formulae in terms of local coordinates involve the so-called **Christoffel symbols**

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}).$$

- Notice that $\Gamma^{\mu}{}_{\nu\lambda} = \Gamma^{\mu}{}_{\lambda\nu}$.
- The Christoffel symbols do not define a tensor. In a neighbourhood of a any $p \in \mathcal{M}$ there is a coordinate system (**normal coordinates**) in which the components of the Christoffel symbols vanish at the point.

The Levi-Civita connection (II)

Explicit coordinate expressions:

- In terms of the Christoffel one defines the components of $\nabla_a v^b$ as

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^\nu{}_{\lambda\mu} v^\lambda.$$

- For a covector ω_a one can deduce:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\nu\mu} \omega_\lambda.$$

- These expressions generalise in an obvious way to higher valence tensors. For example:

$$\nabla_\mu T^\nu{}_{\lambda\rho} = \partial_\mu T^\nu{}_{\lambda\rho} + \Gamma^\nu{}_{\sigma\mu} T^\sigma{}_{\lambda\rho} - \Gamma^\sigma{}_{\lambda\mu} T^\nu{}_{\sigma\rho} - \Gamma^\sigma{}_{\rho\mu} T^\nu{}_{\lambda\sigma}.$$

- The Levi-Civita connection is defined in such a way that $\nabla_a g_{bc} = 0$.

Geodesics

Definition:

- Let v^a denote the tangent vector to a curve $\gamma : I \to \mathcal{M}$, then the curve is a **geodesic** if and only if
$$v^a \nabla_a v^b = f v^b,$$
with f some function of the curve parameter λ.
- In the case $f = 0$, the parameter is called **affine**. An affine parameter is unique up to an affine transformation $\lambda \mapsto a\lambda + b$ for constants a and b.
- A vector field u^a defined a long a curve γ with tangent v^a is said to be **parallelly transported** along γ if $v^a \nabla_a u^b = 0$.

Lie derivatives

Explicit expressions:

- The **Lie derivative** is another type of derivative defined on a manifold.
- It is independent of the metric tensor.
- The Lie derivative measures the change of a tensor as it is transported along the direction prescribed by a vector field v^a and it is denoted by $\mathcal{L}_v$.
- The Lie derivative of a tensor $T^a{}_{bc}$ is given in local coordinates by

$$\mathcal{L}_v T^\mu{}_{\lambda\rho} = v^\sigma \partial_\sigma T^\mu{}_{\lambda\rho} - \partial_\sigma v^\mu T^\sigma{}_{\lambda\rho} + \partial_\lambda v^\sigma T^\mu{}_{\sigma\rho} + \partial_\rho v^\sigma T^\mu{}_{\lambda\sigma},$$

  and can be verified to be a tensor.
- Lie derivatives of other tensors can be defined in an analogous way.

Curvature

Remark:

- In what follows assume that ∇_a is the *Levi-Civita* connection of a metric g_{ab}

Curvature tensors

- The notion of curvature arises in a natural way by considering the *commutator* of covariant derivatives acting on a vector v^a:

$$\nabla_a \nabla_b v^c - \nabla_b \nabla_a v^c = R^c{}_{dab} v^d,$$

  where $R^c{}_{dab}$ is the **Riemann curvature tensor**.
- The corresponding commutator of covariant derivatives for a covector can be found to be

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = -R^d{}_{cab} \omega_d.$$

- Extensions to higher rank tensors are direct.
- In local coordinates (x^μ) one can write

$$R^\mu{}_{\nu\lambda\rho} = \partial_\lambda \Gamma^\mu{}_{\nu\rho} - \partial_\rho \Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\lambda\sigma} \Gamma^\sigma{}_{\nu\rho} - \Gamma^\mu{}_{\rho\sigma} \Gamma^\sigma{}_{\nu\lambda}.$$

Contractions and symmetries of the Riemann tensor

The Ricci and Einstein tensors

- Taking traces of $R^a{}_{bcd}$ one defines the **Ricci tensor** $R_{bd} \equiv R^a{}_{bad}$ and **Ricci scalar** $R \equiv g^{ab}R_{ab}$.
- It is also customary to define the **Einstein tensor**

$$G_{ab} \equiv R_{ab} - \tfrac{1}{2}Rg_{ab}.$$

Symmetries

The Riemann tensor satisfies the following symmetries:

$$R_{abcd} = -R_{bacd},$$
$$R_{abcd} = R_{cdab},$$
$$R_{abcd} + R_{acdb} + R_{adbc} = 0.$$

The last of these identities is known as the **first Bianchi identity**.

Contractions and symmetries (II)

The second Bianchi identity

- In addition the Riemann tensor satisfies a differential identity, the **second Bianchi identity**:

$$\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0.$$

- Contracting twice this identity with the metric shows that $\nabla^a G_{ab} = 0$.

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Introduction

Conceptual framework

- General Relativity is a **relativistic theory of gravity.** It describes the gravitational interaction as a manifestation of the **curvature of spacetime**.
- As it is the case of many other physical theories, General Relativity admits a formulation in terms of an **initial value problem** (**Cauchy problem**) whereby one prescribes the geometry of spacetime at some instant of time and then one purports to reconstruct it from the initial data.
- One has to make sense of what it means to prescribe the geometry of spacetime at an instant of time.
- Also how to reconstruct the spacetime from the data.
- The initial value problem is the core of **mathematical Relativity** —an area of active research with a number of interesting and challenging open problems.

The Einstein field equations (I)

Basic objects:

- General Relativity postulates the existence of a 4-dimensional manifold $\mathcal{M}$, the **spacetime manifold**.
- Point on $\mathcal{M}$ are called **events**.
- $\mathcal{M}$ is endowed with a Lorentzian metric g_{ab} which in these lectures is assumed to have signature $+2$ —i.e. $(-+++)$.

Spacetimes:

- By a **spacetime** it will understood the a pair $(\mathcal{M}, g_{\mu\nu})$ where the metric $g_{\mu\nu}$ satisfies the **Einstein field equations**

$$R_{ab} - \frac{1}{2} R g_{ab} + \lambda g_{ab} = T_{ab}.$$

  These equations show how matter and energy produce curvature of the spacetime.
- λ denotes the so-called **Cosmological constant**.
- T_{ab} is the **energy-momentum tensor** of the matter model.

The Einstein field equations (II)

Conservation equations:

- The conservation of energy-momentum is encoded in the condition

$$\nabla^a T_{ab} = 0.$$

- The conservation equation is consistent with the Einstein field equations as a consequence of the second Bianchi identity:

$$\nabla^a \left(R_{ab} - \tfrac{1}{2} R g_{ab} + \lambda g_{ab} \right) = 0.$$

Test particles:

The geometry of the spacetime can be probed by means of the movement of **test particles**:

- massive test particles move along timelike geodesics;
- rays of light move along null geodesics.

Isolated systems and the vacuum field equations

Some simplifying assumptions:

- Attention will be restricted to the gravitational field of systems describing **isolated bodies**. Henceforth we assume that $\lambda = 0$.
- Moreover, attention is restricted to the **vacuum** case for which $T_{ab} = 0$. The vacuum equations apply in the region external to an astrophysical source, but they usefulness is not restricted to this.
- One of the main properties of the gravitational field as described by General Relativity is that it can be a source of itself —this is a manifestation of the non-linearity of the Einstein field equations.
- This property gives rise to a variety of phenomena that can be analysed by means of the so-called **vacuum Einstein field equations** without having to resort to any further considerations about matter sources:

$$R_{ab} = 0.$$

- The field equations prescribe the geometry of spacetime locally. However, they do not prescribe the topology of the spacetime manifold.

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Solutions to the Einstein field equations

Some conceptual questions:

- Given the vacuum field equations a natural question is whether there are any solutions.
- What should one understand for a solution to the Einstein field equations?

Some first answers:

- In first instance a solution is given by a metric g_{ab} expressed in a specific coordinate system (x^μ) —i.e. $g_{\mu\nu}$. We call this an **exact solution**.
- Exact solutions are our main way of acquiring intuition about the behaviour of generic solutions to the Einstein field equations.

The Minkowski spacetime

In a nutshell:

- The solution is encoded in the line element

$$g = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1,\, 1,\, 1,\, 1).$$

- One clearly verifies that for this metric in these coordinates $R_{\mu\nu\lambda\rho} = 0$ so that $R_{\mu\nu} = 0$.
- As $R_{\mu\nu}$ are the components of a tensor in a specific coordinate system one concludes then $R_{ab} = 0$.
- Any metric related to by a coordinate transformation is a solution to the vacuum field equations.

Observation:

The example in the previous paragraph shows that as a consequence of the tensorial character of the Einstein field equations a solution to the equations is, in fact, an **equivalence class of solutions** related to each other by means of coordinate transformations.

Symmetry assumptions

Motivation:

- In order to find further explicit solutions to the field equations one needs to make some sort of assumptions about the spacetime.
- A standard assumption is that the spacetime has *continuous symmetries*.

Continuous symmetries and Killing vectors

- The notion of a continuous symmetry is formalised by the notion of a **diffeomorphism**.
- A diffeomorphism is a smooth map ϕ of $\mathcal{M}$ onto itself.
- Intuitively the diffeomorphism moves the points in the manifold along curves in the manifold —the **orbits of the symmetry**.
- Let ξ^a denote the tangent vector to the orbits. The mapping ϕ is called an **isometry** if $\mathcal{L}_\xi g_{ab} = 0$. It can be checked that

$$\nabla_a \xi_b + \nabla_b \xi_a = 0.$$

This equation is called the **Killing equation**.

Properties of the Killing equation:

Restrictions on the spacetime

- The Killing equation is **overdetermined** —i.e. it does not admit a solution for a general spacetime.
- Thus a solution, if exists, imposes restrictions on the spacetime.
- Using the commutator

$$\nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c = -R^d{}_{cab}\xi_d,$$

together with the Killing equation one obtains

$$\nabla_a \nabla_b \xi_c = R^d{}_{abc}\xi_d.$$

This is an **integrability condition** for the Killing equation —i.e. a necessary condition that needs to be satisfied by any solution.

Spherical symmetry

Spherical symmetry in a nutshell:

- An important type of symmetry is given by the so-called **spherical symmetry**.
- There exists a 3-dimensional group of symmetries with 2-dimensional spacelike orbits.
- Each orbit is an **homogeneous** and **isotropic** manifold.
- The orbits are required to be compact and to have constant positive curvature.

The Schwarzschild spacetime

The metric in standard coordinates:

- In standard coordinates (t, r, θ, φ) by the expression

$$g = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2).$$

- This solution is **spherically symmetric** and **static** —i.e. time independent.
- The Schwarzschild solution is of particular interest as it gives the simplest example of a **black hole**. The spacetime manifold can be explicitly verified to be singular at $r = 0$. This singularity is hidden behind a **horizon**.

Properties of the Schwarzschild spacetime

Rigidity results:

- The Schwarzschild spacetime satisfies a number of **rigidity properties** —i.e. certain properties about solutions to the Einstein field equations immediately imply other properties.
- Staticity can be obtained from the assumption of spherical symmetry —the **Birkhoff theorem**: any spherically symmetric solution to the vacuum field equations is locally isometric to the Schwarzschild solution
- The Schwarzschild solution can be characterised as the only static solution of the vacuum field equations satisfying a certain (reasonable) behaviour at infinity —**asymptotic flatness**: the requirement that asymptotically, the metric behaves like the Minkowski metric. This result is known as the **no-hair theorem**.

Other exact solutions (I)

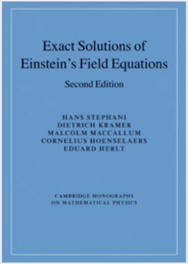
The Kerr spacetime:

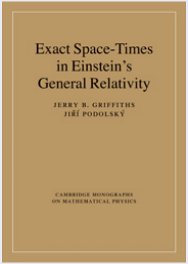
- In order to obtain more exact solutions reduce the number of symmetries —accordingly the task of finding solutions becomes harder.
- A natural assumption is to look for **axially symmetric** and **stationary solutions**.
  - stationarity is a form of time independence which is compatible with the notion of rotation —to be seen in more detail.
- The above assumptions lead to the **Kerr spacetime** describing a time independent rotating black hole.

Other exact solutions (II)

Surveys of exact solutions:

- Although there are a huge number of explicit solutions to the Einstein field equation —see e.g. [Stephani et al], the number of solutions with a physical/geometric relevance is much more restricted.
- For a discussion of some of the physically/geometrically important solutions see e.g. [Griffiths & Podolski].
- For exact solutions describing isolated systems which are time dependent, there are no known solutions without some sort of **pathology**.





Abstract analysis of the Einstein field equations

An alternative to exact solutions:

- Use the general features and structure of the equations to assert existence in an **abstract sense**.
- Proceed in the same way to establish uniqueness and other properties of the solutions.
- In this way can explore more systematically the space of solutions to the theory.
- After this of analysis has been carried out one can proceed to construct solutions **numerically**.

Outline

Introduction:

A strategy:

- A strategy to study generic solutions to the Einstein field equations is to formulate an **initial value problem** (**Cauchy problem**) for the Einstein field equations.
- In order to do so, one needs to bring the equations to some standard form in which the methods of the theory of partial differential equations can be applied.
- One expects the Einstein equations to imply some evolution process.
- Suitable equations describing evolutive processess are **wave equations**.

Outline

The scalar wave equation (I)

The problem:

- On a spacetime $(\mathcal{M}, g_{ab})$ consider the wave equation with respect to the metric g_{ab} —i.e.

$$\square\phi \equiv \nabla_a \nabla^a \phi = 0.$$

- In local coordinates it can be shown that

$$\square\phi = \frac{1}{\sqrt{-\det g}} \partial_\mu \left(\sqrt{-\det g}\, g^{\mu\nu} \partial_\nu \phi \right).$$

The scalar wave equation (I)

The problem:

- On a spacetime $(\mathcal{M}, g_{ab})$ consider the wave equation with respect to the metric g_{ab} —i.e.

$$\Box\phi \equiv \nabla_a \nabla^a \phi = 0.$$

- In local coordinates it can be shown that

$$\Box\phi = \frac{1}{\sqrt{-\det g}}\partial_\mu\left(\sqrt{-\det g}\, g^{\mu\nu}\partial_\nu\phi\right).$$

Principal part:

- The **principal part** of the equation corresponds to the terms containing the highest order derivatives of the scalar field ϕ:

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi.$$

- The structure in this expression is particular of a class of partial differential equations known as **hyperbolic equations**.

The scalar wave equation (II)

The scalar wave in Minkowski spacetime

- The most well known hyperbolic equation is the wave equation on the **Minkowski spacetime**.
- In standard Cartesian coordinates one has that

$$\Box\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = \partial_x^2\phi + \partial_y^2 + \partial_z^2\phi - \partial_t^2\phi = 0.$$

Cauchy problem for the wave equation

- The Cauchy problem for the wave equations and more generally hyperbolic equations is well understood at least in a **local setting**.
- If one prescribes the field ϕ and its derivative $\partial_\mu\phi$ at some fiduciary instant of time $t = 0$, then the equation $\Box\phi = 0$ has a solution for suitably small times (**local existence**).
- This solution is **unique** in its existence interval and it has **continuous dependence** on the initial data.
- The solution exhibits **finite speed propagation**.

Outline

The Maxwell equations (I)

The source free equations:

- A useful model to discuss certain issues arising in the Einstein field equations are the **source-free Maxwell equations**:

$$\nabla^a F_{ab} = 0, \qquad \nabla_{[a} F_{bc]} = 0,$$

  where $F_{ab} = -F_{ab}$ is the **Faraday tensor**.
- A solution to the second Maxwell equation is given by

$$F_{ab} = \nabla_a A_b - \nabla_b A_a,$$

  where A_a is the so-called **gauge potential**.

Gauge freedom:

- The gauge potential does not determine the the Faraday tensor in a unique way as $A_a + \nabla_a \phi$ with ϕ as scalar field gives the same F_{ab}.

The Maxwell equations (II)

An evolution equation for the gauge potential:

- Substituting into the first Maxwell equation one has that

$$
\begin{aligned}
0 &= \nabla^a \left(\nabla_a A_b - \nabla_b A_a \right) \\
&= \nabla^a \nabla_a A_b - \nabla^a \nabla_b A_a .
\end{aligned}
$$

- Using the commutator

$$
\nabla_a \nabla_b A_c - \nabla_b \nabla_a A_c = -R^d{}_{cab} A_d
$$

one concludes that

$$
\nabla^a \nabla_a A_b - \nabla_b \nabla^a A_a - R^a{}_b A_a = 0.
$$

- Under what circumstances one can assert the existence of solutions to the last equation on a smooth spacetime $(\mathcal{M}, g_{ab})$? Note that the principal part is given by:

$$
\partial^\mu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu .
$$

The Maxwell equations (III)

Exploiting the gauge freedom:

- Making the replacement $A_\nu \to A_\nu + \nabla_\nu \phi$, with ϕ chosen such that
$$\nabla^\mu \nabla_\mu \phi = -\nabla^\mu A_\mu \qquad (1)$$
one obtains that
$$\nabla^\mu A_\mu \to \nabla^\mu A_\mu + \nabla^\mu \nabla_\mu \phi = 0.$$
- Equation (1) is to be interpreted as a wave equation for ϕ with source term given by $-\nabla^\mu A_\mu$. One says that the gauge potential is in the **Lorenz gauge** and it satisfies the wave equation
$$\nabla^\mu \nabla_\mu A_\nu = R^\mu{}_\nu A_\mu. \qquad (2)$$
- Equations (1)-(2) are manifestly hyperbolic so that local existence is obtained provided that suitable initial data is provided.
- The initial data consists of ϕ, $\nabla_\mu \phi$, A_ν and $\nabla_\mu A_\nu$ at some initial time.

Outline

The Einstein equations (I)

The EFE in general coordinates:

- Given general coordinates (x^μ), the Ricci tensor R_{ab} can be explicitly written in terms of the components of the metric tensor $g_{\mu\nu}$ and its first and second partial derivatives as

$$R_{\mu\nu} = \frac{1}{2}\sum_{\lambda,\rho=0}^{3}\left(\partial_\lambda\left(g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu}+\partial_\nu g_{\mu\rho}-\partial_\rho g_{\mu\nu}\right)\right)-\partial_\nu\left(g^{\lambda\rho}\partial_\mu g_{\lambda\rho}\right)\right)$$

$$+\frac{1}{4}\sum_{\lambda,\rho,\sigma,\tau=0}^{3}\Big(g^{\sigma\tau}g^{\lambda\rho}\left(\partial_\sigma g_{\rho\tau}+\partial_\rho g_{\sigma\tau}-\partial_\tau g_{\sigma\rho}\right)\left(\partial_\nu g_{\mu\lambda}+\partial_\mu g_{\lambda\nu}-\partial_\lambda g_{\mu\nu}\right)$$

$$-g^{\rho\sigma}g^{\lambda\tau}\left(\partial_\nu g_{\lambda\sigma}+\partial_\lambda g_{\nu\sigma}-\partial_\sigma g_{\nu\lambda}\right)\left(\partial_\sigma g_{\mu\tau}+\partial_\mu g_{\sigma\tau}-\partial_\tau g_{\sigma\mu}\right)\Big),$$

  where $g^{\lambda\rho}=(\det g)^{-1}p^{\lambda\rho}$ with $p^{\lambda\rho}$ polynomials of degree 3 in $g_{\mu\nu}$.
- The vacuum Einstein field equation implies a **second order quasilinear partial differential equations** for the components of the metric tensor.

The Einstein equations (II)

A more useful form of the equations:

- By recalling the formula for the Christoffels symbols in terms of partial derivatives of the metric tensor

$$\Gamma^{\nu}{}_{\mu\lambda} = \frac{1}{2} g^{\nu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}),$$

and by defining

$$\Gamma^{\nu} \equiv g^{\mu\lambda}\Gamma^{\nu}{}_{\mu\lambda},$$

one can rewrite $R_{\mu\nu}$ more concisely as

$$R_{\mu\nu} = -\frac{1}{2} g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} + \nabla_{(\mu}\Gamma_{\nu)} + g_{\lambda\rho} g^{\sigma\tau}\Gamma^{\lambda}{}_{\sigma\mu}\Gamma^{\rho}{}_{\tau\nu} + 2\Gamma^{\sigma}{}_{\lambda\rho} g^{\lambda\tau} g_{\sigma(\mu}\Gamma^{\rho}{}_{\nu)\tau}.$$

The Einstein equations (II)

The principal part of the Einstein equations:

- The principal part of the vacuum Einstein field equation can be readily be identified to be

$$-\frac{1}{2}g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} + \nabla_{(\mu}\Gamma_{\nu)}.$$

The Einstein equations (II)

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Wave coordinates:

- Require the coordinates (x^μ) to satisfy the equation

$$\nabla^\nu\nabla_\nu x^\mu = 0,$$

where the coordinates x^μ are treated as a scalar field over $\mathcal{M}$.

- A direct computation shows that

$$\nabla_\nu x^\mu = \partial_\nu x^\mu = \delta_\nu{}^\mu,$$
$$\nabla_\lambda\nabla_\nu x^\mu = \partial_\lambda\delta_\nu{}^\mu - \Gamma^\rho{}_{\lambda\nu}\delta_\rho{}^\mu = -\Gamma^\mu{}_{\nu\lambda},$$

so that

$$\nabla^\nu\nabla_\nu x^\mu = g^{\nu\lambda}\Gamma^\mu{}_{\nu\lambda} = -\Gamma^\mu.$$

The Einstein equations (III)

Hyperbolic reduction of the equations:

- If suitable initial data is provided for the wave equation $\nabla^\nu \nabla_\nu x^\mu = 0$ —the coordinate differentials $\mathrm{d}x^a$ have to be chosen initially to be point-wise independent— then general theory of hyperbolic differential equations ensures the existence of a solution.
- It follows then that

$$\Gamma^\mu = 0.$$

- The **reduced Einstein field equation** takes the form

$$g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} - 2g_{\lambda\rho}g^{\sigma\tau}\Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\tau\nu} - 4\Gamma^\sigma{}_{\lambda\rho}g^{\lambda\tau}g_{\sigma(\mu}\Gamma^\rho{}_{\nu)\tau} = 0$$

- One obtains a system of **quasilinear wave equations** for the components of the metric tensor $g_{\mu\nu}$.
- The local Cauchy problem with appropriate data is **well-posed** —one can show the existence and uniqueness of solutions and their stable dependence on the data.

The Einstein equations (IV)

Some remarks:

- The system of equations is called the **reduced Einstein field equations** and the procedure a **hyperbolic reduction**.
- For the reduced equation one readily has a developed theory of existence and uniqueness available.
- The introduction of a specific system of coordinates **breaks the tensorial character of the Einstein field equations**.
- Given a solution to the reduced Einstein field equations, the latter will also imply a solution to the actual EFE as long as (x^μ) satisfy the equation $\nabla^\nu \nabla_\nu x^\mu = 0$. This requires some delicate analysis —to be seen later.
- The domain on which the coordinates (x^μ) form a good coordinate system depends on the initial data prescribed and the solution $g_{\mu\nu}$ itself. There is little that can be said *a priori* about the domain of existence of the coordinates.
- The data for the reduced equation consists of a prescription of $g_{\mu\nu}$ and $\partial_\lambda g_{\mu\nu}$ at some initial time $t = 0$. **The next step in our discussion is to understand the meaning of this data**.