

Mathematical problems of General Relativity

Lecture 4

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Outline

1 Energy and momentum in General Relativity

- Basic definitions
- Positivity of energy

2 Symmetries and the initial value problem

3 Epilogue: the propagation of the wave coordinate condition

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Energy and momentum in theories of gravitation: basic issues

The equivalence principle:

- It is a well known problem in General Relativity that energy and momentum of the gravitational field cannot be localised.
- This is a direct consequence of the **equivalence principle**.
- As a consequence one cannot define, for example, a density of energy for the gravitational field.
- However, it is still possible to define some **global conserved quantities** which, in turn, can be interpreted as the total energy of a gravitating system.
- These quantities behave in a similar way to electromagnetic charges —that is, they take the form of volume integrals which are transformed into surface integrals.

The ADM energy and momentum (I)

The definition:

- In what follows let $(\mathcal{S}, h_{ij}, K_{ij})$ denote an initial data set for the vacuum Einstein field equations —i.e. they satisfy the constraints.
- Let x^α denote **asymptotically Cartesian coordinates** —i.e. a system of coordinate for which $h_{\alpha\beta}$ is $\delta_{\alpha\beta}$ to first order.
- One defines the **ADM energy** as the surface integral

$$E = \frac{1}{16\pi} \int_{S_\infty} (\partial^\beta h_{\alpha\beta} - \partial_\alpha) n^\alpha dS, \quad h \equiv h_{\alpha\beta} \delta^{\alpha\beta}.$$

where S_∞ denotes the **sphere at infinity**, and n^α is the outward pointing normal to the sphere. Similarly, the **ADM momentum** is given by

$$p^\alpha = \frac{1}{8\pi} \int_{S_\infty} (K^\alpha{}_\beta - K \delta^\alpha{}_\beta) n^\alpha dS.$$

The ADM energy and momentum (II)

Coordinate independence:

- The expressions can be shown to be coordinate independent.
- In particular, a change to another asymptotically Cartesian system gives the same ADM mass and momentum.
- The energy E and the momentum p^α are the components of a 4-dimensional vector (4-vector) —the **ADM 4-momentum vector**:

$$p^\mu = (E, p^\alpha).$$

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Finiteness:

- If one has an initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ satisfying

$$h_{\alpha\beta} - \delta_{\alpha\beta} = O(1/r), \quad K_{ij} = O(1/r^2),$$

then one can readily verify that

$$E < \infty, \quad p^\alpha < \infty.$$

- The verification of the above statement for p^α makes use of the constraint equations.

The ADM energy and momentum (III)

Some intuition: the Schwarzschild spacetime

- In order to obtain intuition into the content of the ADM energy and momentum, it is convenient to evaluate them on the Schwarzschild spacetime.
- Make use of the time symmetric hypersurface given in standard coordinates by constant t .
- As already seen, for this hypersurface it has been seen that $K_{ij} = 0$. Moreover, one has that

$$h_{\alpha\beta} = \left(1 + \frac{m}{2r}\right)^4 \delta_{\alpha\beta}.$$

- A calculation then shows that

$$E = m, \quad p^\alpha = 0.$$

- The ADM energy of the time symmetric slice of the Schwarzschild spacetime coincides with its mass parameter.

Conservation of the ADM energy and momentum (I)

Intuition:

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Showing the conservation:

- Consider an evolution off the hypersurface \mathcal{S} such that

$$\alpha = 1 + O(1/r), \quad \beta^\alpha = O(1/r).$$

The latter corresponds to an evolution into nearby hypersurfaces \mathcal{S} which are essentially a time translation at infinity. From the above assumptions it follows that $\mathcal{L}_\beta g_{\mu\nu} = O(1/r^4)$.

- One then computes $\partial_t E$ to obtain

$$\partial_t E = \int_{\mathcal{S}_\infty} (\partial_t \partial^\beta h_{\alpha\beta} - \partial_t \partial_\alpha h) n^\alpha dS.$$

- Using the ADM evolution equations one can readily verify by inspection that

$$\partial_t \partial^\beta h_{\alpha\beta} - \partial_t \partial_\alpha h = O(1/r^3) \implies \partial_t E = 0.$$

Conservation of the ADM energy and momentum (II)

Showing the conservation:

- A similar argument shows that $\partial_t p^\alpha = 0$. Thus, indeed, the components of p^μ are conserved, at least for evolutions which behave as a time translation at infinity.

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Positivity of energy in (pseudo) Newtonian theories (I)

Basic intuition:

- On intuitive grounds one would expect the ADM 4-momentum to satisfy some **positivity properties**.
- This is not at all obvious from the definitions in terms of surface integrals of the ADM energy and momentum.

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A toy model:

- Let ϕ denote the **gravitational potential** and let ρ denote the **density of matter**.
- In physically realistic situations one expects ρ to be a function of **compact support**—that is, it vanishes outside a compact set. This requirement fits naturally with the notion of isolated system.
- The gravitational potential is related to the density via the Poisson equation

$$\Delta\phi = 4\pi G\rho.$$

- The total mass of the system is just the integral of the density over the whole space:

$$m = \int_{\mathbb{R}^3} \rho d^3x < \infty.$$

Positivity of energy in (pseudo) Newtonian theories (II)

Computation of the total energy:

- The total energy of the system is then given (using special relativistic arguments) by

$$\begin{aligned}
 E_{total} &= mc^2 + E_{grav} \\
 &= c^2 \int_{\mathbb{R}^3} \rho d^3x + \frac{1}{2} \int_{\mathbb{R}^3} \rho \phi d^3x \\
 &= c^2 \int_{\mathbb{R}^3} \rho d^3x + \frac{1}{8\pi G} \int_{\mathbb{R}^3} \phi \Delta \phi d^3x \\
 &= c^2 \int_{\mathbb{R}^3} \rho d^3x - \frac{1}{8\pi G} \int_{\mathbb{R}^3} |\nabla \phi|^2 d^3x.
 \end{aligned}$$

- In the last equation is negative so that the energy is, in principle, **not bounded from below**.
- This is a problem, as it could mean one could extract an infinite amount of energy out of a gravitating system.
- General Relativity deals with this problem by **postulating the Universality of Gravity**—that is, the fact that gravity can act as source of itself.

Positivity of energy in (pseudo) Newtonian theories (III)

A non-linear theory of Gravity:

- consider a modified theory of gravity which contains a non-linear term:

$$\Delta\phi = 4\pi G\bar{\rho}, \quad \bar{\rho} = \rho - \frac{1}{8\pi Gc^2}|\nabla\phi|^2.$$

- One has that $\bar{\rho}$ is an **effective density** with the non-linear term $|\nabla\phi|^2$ being the gravitational contribution to the mass density.
- In order to analyse the implications of this new theory assume

$$\phi = -\frac{m}{r} + O(1/r^2).$$

- In what follows it is convenient to consider the auxiliary quantity

$$\psi = e^{\phi/2c^2}, \quad \nabla\psi = \frac{1}{2c^2}\psi\nabla\phi, \quad \Delta\psi = \frac{2\pi G}{c^2}\rho\psi.$$

- It can then be verified that

$$4\pi m = \int_{S_\infty} \nabla\psi \cdot \vec{n}dS = \int_{\mathbb{R}^3} \Delta\psi d^3x = \frac{2\pi G}{c^2} \int_{\mathbb{R}^3} \rho\psi d^3x > 0.$$

Energy positivity in General Relativity (I)

Observation:

- A similar mechanism works in General Relativity to ensure the positivity of the energy —the so called **mass positivity theorem**, Schoen & Yau 1979-1981.

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Theorem

Consider a time symmetric initial data set for the vacuum Einstein field equations —i.e. $K_{ij} = 0$. Assume that $\mathcal{S} \approx \mathbb{R}^3$ with

$$h_{\alpha\beta} - \delta_{\alpha\beta} = O(1/r),$$

and that $r_{ij}\lambda^i\lambda^j \geq 0$ for $\lambda^i \neq 0$. Then

$$E > 0.$$

Energy positivity in General Relativity (II)

Harmonic coordinates:

- The proof of the theorem makes use of **harmonic coordinates** $y^\alpha(x)$:

$$\Delta y^\alpha = D_i D^i y^\alpha = 0, \quad y^\alpha = x^\alpha + O(1).$$

The existence of these coordinates is a further assumption which needs to be justified —this, however, will not concern us here.

- It is noticed that

$$|Dy|^2 \equiv h^{ih} D_i y^\alpha D_j y^\alpha, \quad \Delta |Dy|^2 = 2(D_i D_j y^\alpha)(D^i D^j y^\alpha) + 2(D^i y^\alpha)(\Delta D_i y^\alpha),$$

and moreover that

$$\Delta |Dy|^2 = 2|DDy|^2 + 2r_{ij} D^i y^\alpha D^j y^\alpha \geq 0$$

by assumption.

- From the use of harmonic coordinates one has that $\Gamma^\alpha = 0$, which can be reexpressed to show that

$$h^{\beta\gamma}(2\partial_\beta h_{\alpha\gamma} - \partial_\alpha h_{\beta\gamma}) = 0.$$

Energy positivity in General Relativity (III)

The energy as a volume integral:

- The harmonic coordinates can be used to manipulate the expression of the energy:

$$E = \frac{1}{16\pi G} \int_{S_\infty} (\partial_\beta h_{\alpha\beta} - \partial_\alpha h n^\alpha) dS = \frac{1}{32\pi G} \int_{S_\infty} \partial^\alpha h n_\alpha dS.$$

- In the last expression one can one can apply Gauss theorem to get

$$E = \frac{1}{32\pi G} \int_{S_\infty} \partial^\alpha h n_\alpha dS = \frac{1}{32\pi G} \int_S \Delta h d^3y.$$

- Moreover, it can be shown that in the coordinates being used $h^{\beta\beta} = |Dy|^2$. Thus, one concludes

$$E = \frac{1}{32\pi G} \int_S \Delta |Dy|^2 d^3y \geq 0.$$

Rigidity part of the positivity of energy theorem

Remark:

- The theorem has also a **rigidity part**:
 - If the energy vanishes and the hypersurface is regular, then the hypersurface is flat.
- This implies that vacuum cannot gravitate —notice that this result depends strongly on the boundary conditions being used.

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Encoding symmetries in the initial data

The problem:

An issue which often arises in the analysis of the Cauchy problem for the Einstein field equations is that of encoding in the initial data the fact that the resulting spacetime will have a certain symmetry —i.e. a Killing vector. This naturally leads to the notion of **Killing initial data (KID)**.

Some consequences of the Killing equation (I)

Idea:

In order to analyse the question raised in the previous paragraph, it is necessary to first consider some consequences of the Killing equation

$$\nabla_a \xi_b + \nabla_b \xi_a = 0.$$

An integrability condition:

Applying ∇^a to the above equation and commuting covariant derivatives one finds that

$$\begin{aligned} 0 &= \nabla^a \nabla_a \xi_b + \nabla^a \nabla_b \xi_a \\ &= \square \xi_b - R^c{}_b \xi_c, \end{aligned}$$

where it has been used that $\nabla^a \xi_a = 0$. Accordingly, in vacuum one has that a Killing vector satisfies the Killing vector equation

$$\square \xi_a = 0.$$

In what follows a ξ_a satisfying the wave equation (11) will be called a **Killing vector candidate**.

Some consequences of the Killing equation (II)

A wave equation for the Killing equation

Now, in what follows let

$$S_{ab} \equiv \nabla_a \xi_b + \nabla_b \xi_a,$$

and compute $\square S_{ab}$. To this end notice that commuting covariant derivatives and using that by assumption $R_{ab} = 0$ and $\nabla^e R^f{}_{bea} = 0$ one has

$$\begin{aligned} \square S_{ab} &= R^e{}_a{}^f{}_b \nabla_f \xi_e + R^e{}_a{}^f{}_b \nabla_e \xi_f + \nabla_a \square \xi_b + \nabla_b \square \xi_a \\ &= R^e{}_a{}^f{}_b S_{ef} + \nabla_a \square \xi_b + \nabla_b \square \xi_a. \end{aligned}$$

The KID conditions

Obtaining conditions on the initial data:

Assume that one has a vector ξ^a satisfying $\square \xi_a = 0$. One has then that

$$\square S_{ab} - R^e{}_a{}^f{}_b S_{ef} = 0.$$

If initial data on an hypersurface \mathcal{S} can be chosen such that

$$S_{ab} = 0, \quad \nabla_c S_{ab} = 0, \quad \text{on } \mathcal{S}$$

then because of the homogeneity of the wave equation for S_{ab} , it follows that necessarily $S_{ab} = 0$ in the development of \mathcal{S} so that ξ^a is, in fact, a Killing vector. The conditions are called the **Killing initial Data (KID) conditions**.

Intrinsic conditions (I)

A 3+1 split:

The KID equations are conditions not only on ξ^a but also on the initial data $(\mathcal{S}, h_{ij}, K_{ij})$. Writing

$$\xi^a = Nn^a + N^a, \quad n_a N^a = 0,$$

where N and N^a denote the **lapse and shift of the Killing vector**, a computation then shows that the space-space components of the Killing equation $\nabla_a \xi_b + \nabla_b \xi_a = 0$ imply

$$NK_{ij} + D_{(i} Y_{j)} = 0.$$

Taking a time derivative of the above equation and using the ADM evolution equations one finds that

$$N^k D_k K_{ij} + D_i N^k K_{kj} + D_j N^k K_{ik} + D_i D_j N = N(r_{ij} + K K_{ij} - 2K_{ik} K^k_j).$$

Intrinsic conditions (II)

Theorem

Let $(\mathcal{S}, h_{ij}, K_{ij})$ denote an initial data set for the vacuum Einstein field equations. If there exists a pair (N, N^i) such that

$$NK_{ij} + D_{(i}Y_{j)} = 0,$$

$$N^k D_k K_{ij} + D_i N^k K_{kj} + D_j N^k K_{ik} + D_i D_j N = N(r_{ij} + K K_{ij} - 2K_{ik} K^k_j),$$

then the development of the initial data has a Killing vector.

Remarks:

- The KID conditions are overdetermined. This is natural as not every spacetime admits a symmetry.
- The KID conditions are closely related to the constraint equations and the ADM evolution equations. This is a deep relation which will not be explored here!

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The reduced Einstein equations vs the Einstein equations

Some remarks:

- At the beginning of the lectures it was shown that if the coordinates satisfy the wave coordinate condition $\square x^\mu = 0$, then the Einstein field equations reduce to a system of quasilinear wave equations for the components of the metric $g_{\mu\nu}$ —the so-called **reduced Einstein equations**.
- To conclude the discussion it is now shown that under suitable conditions the Einstein reduced equations imply a solution of the actual Einstein field equations.
- This in fact, is equivalent to showing that if the contracted Christoffel symbols $\Gamma^\mu \equiv g^{\nu\lambda}\Gamma^\mu_{\nu\lambda}$ vanish initially, then they also vanish at any later time.

Propagating the wave coordinate condition

A computation:

The key observation is that the *reduced Einstein field equations* can be written as

$$R_{\mu\nu} = \nabla_{(\mu}\Gamma_{\nu)}.$$

Now, using the contracted Bianchi identity

$$\nabla^\mu (R_{\mu\nu} - \frac{1}{2}Rg_\mu) = 0,$$

it follows that

$$\square\Gamma_\mu + R^\nu{}_\mu Q_\mu = 0.$$

This is a wave equation for the contracted Christoffel symbol. In view of its homogeneity, if

$$\Gamma_\mu = 0, \quad \nabla_\nu\Gamma_\mu = 0, \quad \text{on } \mathcal{S}$$

then $\Gamma_\mu = 0$ in the development of \mathcal{S} and accordingly $R_{\mu\nu} = 0$.

Remark:

The initial conditions on the contracted Christoffel symbols are related to the constraint equations.