2
Differential Geometry

The language of General relativity is Differential Geometry. The preset chapter provides a brief review of the ideas and notions of Differential Geometry that will be used in the book. In this role, it also serves the purpose of setting the notation and conventions to be used througout the book. The chapter is not intended as an introduction to Differential Geometry and assumes a prior knowledge of the subject at the level, say, of the first chapter of Choquet-Bruhat (2008) or Stewart (1991), or chapters 2 and 3 of Wald (1984). As such, rigorous definitions of concepts are not treated in depth —the reader is, in any case, refered to the literature provided in the text, and that given in the final section of the chapter. Instead, the decision has been taken of discussing at some length topics which may not be regarded as belonging to the standard bagage of a relativist. In particular, some detail is provided in the discussion of general (i.e. non Levi-Civita) connections, the sometimes $1+3$ split of tensors (i.e. a split based on a congruence of timelike curves, rather than on a foliation as in the usual $3+1$), and a discussion of submanifolds using a frame formalism. A discussion of conformal geometry has been left out of this chapter and will be undertaken in Chapter 5.

2.1 Manifolds

The basic object of study in Differential Geometry is a *differentiable manifold* . Intuitively, a manifold is a space that locally looks like $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Despite this simplicity at a small scale, the global structure of a manifold can be much more complicated and leads to considerations of Differential Topology.

2.1.1 On the definition of a manifold

Before further discussing the notion of a differentiable manifold, some terminology is briefly recalled.

A function f between open sets of $\mathcal{U}, \mathcal{V} \in \mathbb{R}^n$, $f : \mathcal{U} \to \mathcal{V}$, is called a ***diffeomorphism*** if it is bijective and if its inverse $f^{-1} : \mathcal{V} \to \mathcal{U}$ is differentiable. If f and f^{-1} are C^k functions, then one has a ***C^k-diffeomorphism***. Furthermore, if f and f^{-1} are C^∞ functions, one talks of a ***smooth diffeomorphism***. Althroughout this book, the word ***smooth*** will be used as a synonym of C^∞. The words function, map and mapping will also be used as synonyms of each other.

Provide a picture of what follows as in [Ste91]

A ***topological space*** is a set with a well defined notion of open an closed sets. Given some topological space $\mathcal{M}$, a ***chart*** on $\mathcal{M}$ is a pair $(\mathcal{U}, \varphi)$, with $\mathcal{U} \subset \mathcal{M}$ and φ a bijection from $\mathcal{U}$ to an open set $\varphi(\mathcal{U}) \subset \mathbb{R}^n$ such that given $p \in \mathcal{U}$

$$\varphi(p) \equiv (x^1, \ldots, x^n).$$

The entries $x^1, \ldots, x^n$ are called ***local coordinates*** of the point $p \in \mathcal{M}$. The set $\mathcal{U}$ is called the ***domain*** of the chart. Two charts $(\mathcal{U}_1, \varphi_1)$ and $(\mathcal{U}_2, \varphi_2)$ are said to be ***C^k-related*** if the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2),$$

and its inverse are C^k. The map $\varphi_1 \circ \varphi_2^{-1}$ define changes of local coordinates, $(x^\mu) = (x^1, \ldots, x^n) \mapsto (y^\mu) = (y^1, \ldots, y^n)$, in the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$. In this sense, one can regard the coordinates (y^μ) as functions of the coordinates (x^μ). *All throughout this book the greek letters μ, $\nu, \ldots$ will be used to denote coordinate indices.* The functions $y^a(x^1, \ldots, x^n)$ are C^k and, moreover, the ***Jacobian*** , $\det(\partial y^\mu / \partial x^\nu)$, is different from zero.

A ***C^k-atlas*** on $\mathcal{M}$ is a collection of charts, whose domains cover the set $\mathcal{M}$. The collection of all C^k-related charts is called a ***maximal atlas*** . The pair consisting of the space $\mathcal{M}$ together with its maximal C^k-atlas is called a ***C^k-differentiable manifold***. If the charts are C^∞ related, one speaks of a ***smooth differentiable manifold***. If for each each φ in the atlas the map $\varphi : \mathcal{U} \to \mathbb{R}^n$ has the same n, then the manifold is said to have ***dimension*** n . *The discussion of this book will only be concerned with manifolds of dimension 3 and 4.*

Remark. Introductory discussions of Differential Geometry are generally carried out under the assumption of smooth structures. However, as it will be seen in later chapters, when one looks at General Relativity from the perspective of Conformal Geometry, the smoothness (or lack thereof) acquires an important physical content. Accordingly, one is lead to consider the more general class of C^k-differentiable manifolds.

The differential manifolds used in General Relativity are generally assumed to be Hausdorff and paracompact. A differentiable manifold is ***Hausdorff*** if every two points in it admit non-interseting open neighbourhoods. The reason for requiring he Hausdorff condition is to ensure that a convergent sequence of points cannot have more than one limit point. If $\mathcal{M}$ is ***paracompact*** then there exists a countable basis of open sets. Paracompactness is used in several constructions in Differential Geometry. In particular, it is required to show that every manifold admits a metric. *In what follows, all differential manifolds to be considered will be assumed to be Hausdorff and paracompact.* In the rest of the book Hausdorff, paracompact differential manifolds will be simply called ***manifolds***.

Orientability

A desirable property of the manifolds generally considered in General Relativity is orientability. An open set of $\mathbb{R}^n$ is naturally oriented by the order of the coordinates $(x^\mu) = (x^1, \ldots, x^n)$. Hence, a chart $(\mathcal{U}, \varphi)$ inherits an orientation from its image in $\mathbb{R}^n$. In an orientable manifold the orientation of these charts match together properly. More precisely, a manifold is said to be ***orientable*** if its maximal atlas is such that the Jacobian of the coordinate transformation for each pair of overlapping charts is positive . An alternative description of the notion of orientability in terms of orthonormal frames of vectors will be given in Section 2.5.3. Orientability is a necessary and sufficient condition for the existence of certain structures on $\mathcal{M}$ —see e.g. 3.

2.1.2 Manifolds with boundary

Manifolds with boundary arise naturally when discussing General Relativity from a perspective of Conformal Geometry. In order to introduce this concept one requires the following subsets of $\mathbb{R}^n$:

$$\mathbb{H}^n \equiv \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\},$$
$$\partial\mathbb{H}^n \equiv \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid x^n = 0\}.$$

One says that $\mathcal{M}$ is a ***manifold with boundary*** if $\mathcal{M}$ if it can be covered with charts mapping open subsets of $\mathcal{M}$ either to open sets of $\mathbb{R}^n$ or to open subsets of $\mathbb{H}^n$. The ***boundary of*** $\mathcal{M}$, $\partial\mathcal{M}$, is the set of points $p \in \mathcal{M}$ for which there is a chart $(\mathcal{U}, \varphi)$ with $p \in \mathcal{U}$ such that $\varphi(\mathcal{U}) \subset \mathbb{H}^n$

and $\varphi(p) \in \partial\mathbb{H}^n$. The boundary $\partial\mathcal{M}$ is an $n-1$ dimensional differentiable manifold in its own right. Hence, it is a *submanifold* of $\mathcal{M}$ —see Section 2.7.1.

2.2 Vectors and tensors on a manifold

2.2.1 Some ancillary notions

This section begins by briefly recalling some notions which play a role in the discussion of vectors and tensors on a manifold.

Derivations

Denote by $\mathfrak{X}(\mathcal{M})$ the set of ***scalar fields (i.e. functions) over*** $\mathcal{M}$, i.e. smooth functions $f : \mathcal{M} \to \mathbb{R}$. The set $\mathfrak{X}(\mathcal{M})$ is a *commutative algebra* with respect to pointwise addition and multiplication.

Definition *A derivation is a map* $\mathcal{D} : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ *such that:*

(a) **Action on constants.** *For all constant fields* k, $\mathcal{D}(k) = 0$.
(b) **Linearity.** *For all* f, $g \in \mathfrak{X}(\mathcal{M})$, $\mathcal{D}(f+g) = \mathcal{D}(f) + \mathcal{D}(g)$;
(c) **Leibnitz rule.** *For all* f, $g \in \mathfrak{X}(\mathcal{M})$, $\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g)$.

The connection between derivations and covariant derivatives will be briefly discussed in Section (2.4.1).

Curves

The notion of a vector is intimately related to that of a curve. Given an open interval $I = (a, b) \subset \mathbb{R}$ where either or both of a, b can be infinite, a smooth ***curve*** on $\mathcal{M}$ is a map $\gamma : I \to \mathcal{M}$ such that for any chart $(\mathcal{U}, \varphi)$, the composition $\varphi \circ \gamma : I \to \mathbb{R}^n$ is a smooth map. One very often speaks of the curve $\gamma(t)$ with $t \in (a, b)$.

A ***tangent vector to a curve*** $\gamma(t)$ at a point $p \in \mathcal{M}$, $\dot{\gamma}(p)$, is the map defined by

$$\dot{\gamma}(p) : f \mapsto \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(p) = \dot{\gamma}(f)(p),$$

where $f \in \mathfrak{X}(\mathcal{M})$. Given a chart $(\mathcal{U}, \varphi)$ with local coordinates (x^μ), the components of $\dot{\gamma}(p)$ with respect to the chart are given by $\dot{x}^\mu(p) \equiv \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\gamma(t))|_p$. In a slight abuse of notation the curve γ will be denoted by $\boldsymbol{x}(t)$ and its tangent vector by $\dot{\boldsymbol{x}}(t)$.

2.2.2 Tangent vectors and covectors

At every point $p \in \mathcal{M}$, one can associate a a vector space $T|_p(\mathcal{M})$, ***the tangent space at*** p. The elements of this space are known as ***vectors***. All throughout, vectors will mostly be denoted with lower-case bold-face latin letters: $\boldsymbol{v}$, $\boldsymbol{u}$, $\boldsymbol{w}$, ... Abstract index notation will also be used to denote vectors —see Section 2.2.6. The tangent space $T|_p(\mathcal{M})$ can be characterised either as the set of derivations at p of smooth functions on $\mathcal{M}$ or as the set of equivalence classes of curves through p under a suitable equivalence relation. With the first characterisation one regards the vectors as *directional derivatives* while with the second as *velocities*. If the dimension of the manifold $\mathcal{M}$ is n, then $T|_p(\mathcal{M})$ is a vector space of dimension n. Local coordinates (x^μ) in a neighbourhood of the point p give a basis of $T|_p(\mathcal{M})$ consisting of the partial derivative operators $\{\boldsymbol{\partial}/\boldsymbol{\partial} x^\mu\}$. Where no confusion arise about which coordinates are meant, one writes simply $\{\boldsymbol{\partial}_\mu\}$. In particular, for the vector tangent to a curve one has that $\dot{\boldsymbol{x}}(t) = x^\mu(t)\boldsymbol{\partial}_\mu$. In this last expression and in what follows, ***Einstein's summation covention*** has been adopted —that is, repeated up and down coordinate indices indicate summation for all values the range of the index. That is,

$$x^\mu(t)\boldsymbol{\partial}_\mu = \sum_{\mu=1}^{n} x^\mu(t)\boldsymbol{\partial}_\mu.$$

Covectors

The ***dual space*** $T^*|_p(\mathcal{M})$ is the vector space of linear maps $\boldsymbol{\omega} : T|_p(\mathcal{M}) \to \mathbb{R}$. Save for some exceptions, elements of $T^*|_p(\mathcal{M})$ will be denoted by lower-case bold-face Greek letters. Being dual to $T|_p(\mathcal{M})$, the space $T^*|_p(\mathcal{M})$ has also dimension n, and its elements are called ***covectors*** or ***1-forms***. If $\boldsymbol{\omega}$ acts on $\boldsymbol{v} \in T|_p(\mathcal{M})$, then one writes $\langle \boldsymbol{\omega}, \boldsymbol{v} \rangle \in \mathbb{R}$.

Given $f \in \mathfrak{X}(\mathcal{M})$, for each $\boldsymbol{v} \in T|_p(\mathcal{M})$, one has that $\boldsymbol{v}(f)$ is a scalar. Hence, f defines a map, the ***differential*** of f, $\mathbf{d}f : T|_p(\mathcal{M}) \to \mathbb{R}$ via

$$\mathrm{d}f(\boldsymbol{v}) = \boldsymbol{v}(f).$$

As a consequence of the linearity of $\boldsymbol{v}$ one also has that $\mathrm{d}f$ is linear, and thus $\mathrm{d}f \in T^*|_p(\mathcal{M})$. Given a chart $(\mathcal{U}, \varphi)$ with coordinates (x^μ), the coordinate differentials $\mathbf{d}x^\mu$ form a basis for $T^*|_p(\mathcal{M})$, the so-called ***dual basis*** . The dual basis satisfies $\langle \mathbf{d}x^\mu, \boldsymbol{\partial}_\nu x \rangle = \delta_\mu{}^\nu$, where $\delta_\mu{}^\nu$ is the so-called ***Kronecker's delta*** . As a consequence, every covector $\boldsymbol{\omega}$ at $p \in \mathcal{M}$ can be written as $\boldsymbol{\omega} = \langle \boldsymbol{\omega}, \boldsymbol{\partial}_\mu \rangle \mathbf{d}x^\mu$.

Bases

The previous discussion is extended in a natural way to more general bases. Given any basis $\{\boldsymbol{e_a}\}$ of $T|_p(\mathcal{M})$, its ***dual basis*** $\{\boldsymbol{\omega^b}\}$ $T^*|_p(\mathcal{M})$ is defined by the condition $\langle \boldsymbol{\omega^b}, \boldsymbol{e_a}\rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$. In the last expression, as well as in the rest of the book, bold-face lower-case indices like $\boldsymbol{a}$, $\boldsymbol{b}, \ldots$ denote ***spacetime frame indices*** ranging $\mathbf{0}, \ldots \mathbf{3}$. These will be used when working 4-dimensional manifolds. In the sequel, ***spatial frame indices*** denoted by lower-case boldface Greek letters as in $\boldsymbol{\alpha}$, $\boldsymbol{\beta}, \ldots$, ranging either $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ or $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ will be used in the context of 3-dimensional manifolds. *For simplicity of presentation, and unless explicitly stated, a 4-dimensional manifold will be assumed in the subsequent discussion.*

Given another bases $\{\tilde{\boldsymbol{e}}_{\boldsymbol{a}}\}$ and $\{\tilde{\boldsymbol{\omega}}^{\boldsymbol{b}}\}$ of, respectively, $T|_p(\mathcal{M})$ and $T^*|_p(\mathcal{M})$, these are related to the bases $\{\boldsymbol{e_a}\}$ and $\{\boldsymbol{\omega^a}\}$ by non-singular matrices $(A_{\boldsymbol{a}}{}^{\boldsymbol{b}})$ and $(A^{\boldsymbol{a}}{}_{\boldsymbol{b}})$ such that

$$\tilde{\boldsymbol{e}}_{\boldsymbol{a}} = A_{\boldsymbol{a}}{}^{\boldsymbol{b}} \boldsymbol{e_b}, \qquad \tilde{\boldsymbol{\omega}}^{\boldsymbol{a}} = A^{\boldsymbol{a}}{}_{\boldsymbol{b}} \boldsymbol{\omega^b}, \tag{2.1}$$

satisfying $A^{\boldsymbol{a}}{}_{\boldsymbol{b}} A^{\boldsymbol{b}}{}_{\boldsymbol{c}} = \delta_{\boldsymbol{c}}{}^{\boldsymbol{a}}$ so that $(A_{\boldsymbol{a}}{}^{\boldsymbol{b}})$ and $(A^{\boldsymbol{a}}{}_{\boldsymbol{b}})$ are inverses of each other. In these last expressions and in what follows, ***Einstein's summation convention*** for repeated contravariant and covariant *frame* indices has been adopted so that a sum from $\boldsymbol{b} = \mathbf{0}$ to $\boldsymbol{b} = \mathbf{3}$ is implied.

Condition (2.1) ensures that the new bases $\tilde{\boldsymbol{e}}_{\boldsymbol{a}}$ and $\boldsymbol{\omega^b}$ are dual to each other —i.e. $\langle \tilde{\boldsymbol{\omega}}^{\boldsymbol{b}}, \tilde{\boldsymbol{e}}_{\boldsymbol{a}}\rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$. Given $\boldsymbol{v} \in T|_p(\mathcal{M})$, $\boldsymbol{\alpha} \in T^*|_p(\mathcal{M})$, the above transformation rules for the bases imply

$$\begin{aligned} \boldsymbol{v} &= v^{\boldsymbol{a}} \boldsymbol{e_a} = \tilde{\boldsymbol{v}}^a \tilde{\boldsymbol{e}}_{\boldsymbol{a}} = (\tilde{v}^{\boldsymbol{a}} A_{\boldsymbol{a}}{}^{\boldsymbol{b}}) \boldsymbol{e_b}, \\ \boldsymbol{\alpha} &= \alpha_{\boldsymbol{a}} \omega^{\boldsymbol{a}} = \tilde{\boldsymbol{\alpha}}_{\boldsymbol{a}} \tilde{\boldsymbol{\omega}}^{\boldsymbol{a}} = (\tilde{\boldsymbol{\alpha}}_{\boldsymbol{a}} A^{\boldsymbol{a}}{}_{\boldsymbol{b}}) \boldsymbol{\omega^b}. \end{aligned}$$

The two bases are said to have the ***same orientation*** if $\det(A_{\boldsymbol{a}}{}^{\boldsymbol{b}}) > 0$. Having the same orientation defines equivalence relation between bases with two equivalence classes.

2.2.3 Higher rank tensors

Higher rank tensors can be constructed using elements of $T|_p(\mathcal{M})$ and $T^*|_p(\mathcal{M})$ as basic building blocks. A ***contravariant tensor of rank*** k at the point p is a mutilinear map

$$\boldsymbol{M} : \underbrace{T^*|_p(\mathcal{M}) \times \cdots T^*|_p(\mathcal{M})}_{k \text{ terms}} \longrightarrow \mathbb{R},$$

that is, a function taking as argument k covectors. Similarly, a ***a covariant tensor of rank*** l at the point p is a multilinear map

$$\boldsymbol{N} : \underbrace{T|_p(\mathcal{M}) \times \cdots T|_p(\mathcal{M})}_{l \text{ terms}} \longrightarrow \mathbb{R},$$

that is, a function taking as arguments l vectors. More generally one can also have ***tensors of mixed type*** . More precisely, a (k,l) tensor at p is a multilinear map

$$\boldsymbol{T} : \underbrace{T^*|_p(\mathcal{M}) \times \cdots T^*|_p(\mathcal{M})}_{k \text{ terms}} \times \underbrace{T|_p(\mathcal{M}) \times \cdots T|_p(\mathcal{M})}_{l \text{ terms}} \longrightarrow \mathbb{R},$$

so that $\boldsymbol{T}$ takes as arguments k covectors and l vectors. In particular, a $(k,0)$-tensor corresponds to a contravariant tensor of rank k while a $(0,l)$-tensor is a covariant tensor of rank l. The ***space of*** (k,l)***-tensors at the point*** p will be denoted by $T^k_l|_p(\mathcal{M})$. In particular, one has the identifications $T^1|_p(\mathcal{M}) = T|_p(\mathcal{M})$ and $T_1|_p(\mathcal{M}) = T^*|_p(\mathcal{M})$. Formally, the space $T^k_l|_p(\mathcal{M})$ is obtained as the tensor product of k copies of $T^*|_p(\mathcal{M})$ and l copies of $T|_p(\mathcal{M})$. That is, one has that

$$T^k_l|_p(\mathcal{M}) = \underbrace{T|_p(\mathcal{M}) \otimes \cdots \otimes T|_p(\mathcal{M})}_{k \text{ terms}} \otimes \underbrace{T^*|_p(\mathcal{M}) \otimes \cdots \otimes T^*|_p(\mathcal{M})}_{l \text{ terms}}.$$

The ordering given in the previous expression is known as the ***standard order*** . Notice, however, that an arbitrary tensor needs not to have their arguments in standard order.

As an example of the previous discussion consider $\boldsymbol{v} \in T|_p(\mathcal{M})$ and $\boldsymbol{\alpha} \in T^*|_p(\mathcal{M})$. Their ***tensor product*** $\boldsymbol{v} \otimes \boldsymbol{\alpha}$ is then defined by

$$(\boldsymbol{v} \otimes \boldsymbol{\alpha})(\boldsymbol{u}, \boldsymbol{\beta}) = \langle \boldsymbol{\beta}, \boldsymbol{v} \rangle \langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle, \qquad \boldsymbol{u} \in T|_p(\mathcal{M}), \qquad \boldsymbol{\beta} \in T^*|_p(\mathcal{M}). \quad (2.2)$$

Following the definitions given in the previous paragraphs one readily sees that $\boldsymbol{v} \otimes \boldsymbol{\alpha}$ is indeed a multilinear map and thus a $(1,1)$-tensor at $p \in \mathcal{M}$. The action of a tensor product given in equation (2.2) can be extended directly to an arbitrary (finite) number of tensors and 1-forms. If $\{\boldsymbol{e_a}\}$ and $\{\boldsymbol{\omega^b}\}$ denote, respectively, bases of $T|_p(\mathcal{M})$ and $T^*|_p(\mathcal{M})$, then a basis of $T^k_l|_p(\mathcal{M})$ is given by

$$\{\boldsymbol{e}_{\boldsymbol{b}_1} \otimes \cdots \otimes \boldsymbol{e}_{\boldsymbol{b}_k} \otimes \boldsymbol{\omega}^{\boldsymbol{a}_1} \otimes \cdots \otimes \boldsymbol{\omega}^{\boldsymbol{a}_l}\}.$$

The collection of all the tensor spaces of the form $T^k_l|_p(\mathcal{M})$ is called the ***tensor algebra*** at p and will be denoted by $T^\bullet|_p(\mathcal{M})$. The tensor algebra is defined by means of a *direct sum*.

Symmetries of tensors

A covariant tensor of rank l $\boldsymbol{N}$ is said to be ***symmetric*** with respect to its i and j arguments if

$$\boldsymbol{N}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_i,\ldots,\boldsymbol{v}_j,\ldots,\boldsymbol{v}_l)=\boldsymbol{N}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_j,\ldots,\boldsymbol{v}_i,\ldots,\boldsymbol{v}_l). \tag{2.3}$$

Similarly, it is said to be ***antisymmetric*** if

$$\boldsymbol{N}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_i,\ldots,\boldsymbol{v}_j,\ldots,\boldsymbol{v}_l)=-\boldsymbol{N}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_j,\ldots,\boldsymbol{v}_i,\ldots,\boldsymbol{v}_l). \tag{2.4}$$

If the property (2.3), respectively (2.4), holds under interchange of any arbitrary pair of indices one says that $\boldsymbol{N}$ is ***totally symmetric***, respectively ***totally antisymmetric***. The above definitions can be extended to contravariant tensors of arbitrary rank. A totally antisymmetric covariant tensor of rank l s called a ***l-form*** . Symmetry properties of tensors are best expressed in terms of abstract index notation.

2.2.4 Tensor fields

The discussion in the previous subsections concerned the notion of a tensor at a point $p\in\mathcal{M}$. The ***tensor bundle*** over $\mathcal{M}$, $\mathfrak{T}^\bullet(\mathcal{M})$ is the (disjoint) union of the tensor algebras $T^\bullet|_p(\mathcal{M})$ for all $p\in\mathcal{M}$:

$$\mathfrak{T}^\bullet(\mathcal{M})\equiv\bigcup_{p\in\mathcal{M}}T^\bullet|_p(\mathcal{M}).$$

The modifier *disjoint* is used in this context to emphasise that although for p, $q\in\mathcal{M}$, $p\neq q$, the spaces $T^\bullet|_p(\mathcal{M})$ and $T^\bullet|_q(\mathcal{M})$ are *isomorphic*, they are regarded as different sets. Important subsets of the tensor bundles are the ***tangent bundle*** and the ***cotangent bundle*** given, respectively, by

$$T(\mathcal{M})\equiv\bigcup_{p\in\mathcal{M}}T_p(\mathcal{M}),\qquad T^*(\mathcal{M})\equiv\bigcup_{p\in\mathcal{M}}T^*|_p(\mathcal{M}).$$

A ***smooth tensor field*** over $\mathcal{M}$ is a prescription of a tensor $\boldsymbol{T}\in T^\bullet|_p(\mathcal{M})$ at each $p\in\mathcal{M}$ such that when $\boldsymbol{T}$ is represented locally in a system of coordinates aroun p, the corresponding functions are smooth functions on the local chart and more generally across the atlas. This idea can be naturally extended to the C^k case. An important property of tensor fields is that they are multilinear over $\mathfrak{X}(\mathcal{M})$. This property is often refered as to ***$\mathfrak{X}$-linearity*** . It can be used to characterise tensors. More precisely, one has the following lemma which will be used repeatedly:

Lemma 1 (Tensor characterisation) *A map*

$$\boldsymbol{T} : T^*(\mathcal{M}) \times \cdots \times T^*(\mathcal{M}) \times T(\mathcal{M}) \times \cdots T(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$$

is induced by a (k,l)*-tensor fied if and only if it is multilinear over* $\mathfrak{X}(\mathcal{M})$*.*

The discussion of tensor fields and the tensor bundle is naturally carried out using the language of *fibre bundles*. This point of view will not, however, be used in this book.

2.2.5 The commutator of vector fields

Given $\boldsymbol{u}$, $\boldsymbol{v} \in T(\mathcal{M})$, their ***commutator*** $[\boldsymbol{u}, \boldsymbol{v}] \in T(\mathcal{M})$ is the vector field defined by

$$[\boldsymbol{u}, \boldsymbol{v}]f \equiv \boldsymbol{u}(\boldsymbol{v}(f)) - \boldsymbol{v}(\boldsymbol{u}(f)),$$

for $f \in \mathfrak{X}(\mathcal{M})$. Given a basis $\boldsymbol{e}_{\boldsymbol{a}}$ one has that the components of the commutator with respect to this basis are given by

$$[\boldsymbol{u}, \boldsymbol{v}]^{\boldsymbol{a}} = \boldsymbol{u}(v^{\boldsymbol{a}}) - \boldsymbol{u}(v^{\boldsymbol{a}}).$$

One can readily verify that

$$\begin{aligned}
&[\boldsymbol{u}, \boldsymbol{v}] = -[\boldsymbol{v}, \boldsymbol{u}], \\
&[\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w}] = [\boldsymbol{u}, \boldsymbol{w}] + [\boldsymbol{v}, \boldsymbol{w}], \\
&[[\boldsymbol{u}, \boldsymbol{v}], \boldsymbol{w}] + [[\boldsymbol{v}, \boldsymbol{w}], \boldsymbol{u}] + [[\boldsymbol{w}, \boldsymbol{u}], \boldsymbol{v}] = 0.
\end{aligned}$$

The last identity is known as the ***Jacobi identity*** —not to be confused with the Jacobi identity for spinors to be discussed in Chapter 3.

2.2.6 Abstract index notation for tensors

The presentation of tensors in this section has used what is usually called an ***index free notation*** . In the sequel, the so-called ***abstract index notation*** will also be used where convenient —see Penrose and Rindler (1984). To this end, lower case latin indices will be employed. Accordingly, a vector field $\boldsymbol{v} \in T(\mathcal{M})$ will also be written as v^a. Similarly, for $\boldsymbol{\alpha} \in T^*(\mathcal{M})$ one writes α_a. More generally, a (k,l)-tensor tensor $\boldsymbol{T}$ will be denoted by $T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$. It is important to stress that the indices in these expressions does not represent components with respect to some coordinates or frame. These are denoted, respectively, by Greek indices and boldface lower-case latin indices like in $v^{\boldsymbol{a}}$ and v^μ. The role of the abstract indices is to specify in a convenient way the nature of the object under consideration and also to describe in a convenient fashion operations between tensors. It is recalled, in particular, that the action $\langle \boldsymbol{\alpha}, \boldsymbol{v} \rangle$ of a 1-form on a vector is denoted in abstract index notation by $\alpha_a v^a$, while its tensor product $\boldsymbol{\alpha} \otimes \boldsymbol{v}$ is written as $\alpha_a v^b$. Similarly the operation defined in equation (2.2) is expressed as $\alpha_a u^a \beta_b v^b$.

As emphasised in Wald (1984), the idea behind the abstract index notation is to have a notation for tensorial expressions that mirrors the expressions for their basis components (had a basis been introduced). Unsing the index notation one can only write down tensorial expressions since no basis has been specified. The distinction is more of spirit than of substance. Each type of notation has its own advantages. In particular, the index free notation is better to describe conceptual and structural aspects, while the abstract index notation is most convenient for performing computations. In particular, the abstract index notation allows to express in a convenient way tensors whose arguments are not given in standard order like in $F_{ab}{}^c{}_d$. An operation which has a convenient description in terms of abstract indices is the ***contraction*** between a contravariant a covariant index. For example, given $F_{ab}{}^c{}_d$, the contraction between the contravariant index ^{c} and, say, the covariant index ${}_d$ is denoted by $F_{ab}{}^c{}_c$. Following the convention that repeated indices are *dummy* one has, for example, that $F_{ab}{}^c{}_c = F_{ab}{}^d{}_d$. If $F_{\boldsymbol{ab}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} \equiv S_{ab}{}^c{}_d \boldsymbol{\omega}^{\boldsymbol{a}} \boldsymbol{\omega}^{\boldsymbol{b}} \boldsymbol{e}_{\boldsymbol{c}} \boldsymbol{\omega}^{\boldsymbol{d}}$ denotes the components of $F_{ab}{}^c{}_d$ with respect to a basis $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ and its associated cobasis $\boldsymbol{\omega}^{\boldsymbol{a}}$, then the components of the contraction $F_{ab}{}^c{}_c$ are given by $F_{\boldsymbol{ab}}{}^{\boldsymbol{c}}{}_{\boldsymbol{c}}$. *Following the Einstein summation convention, a sum on the index* $\boldsymbol{c}$ *is understood.* Despite the need of the need of defining the operation in terms of components with respect to a basis, the contraction is a *geometric* (i.e. coordinate and base independent) operation transforming a tensor of rank (k, l) into a tensor of rank $(k-1, l-1)$.

Symmetries of tensors are expressed in a convenient fashion using abstract index notation. For example, if S_{ab} and A_{ab} denote, respectively symmetric and antisymmetric covariant tensors of rank 2, then $S_{ab} = S_{ba}$ and $A_{ab} = -A_{ba}$. More generally, given M_{ab}, its ***symmetric part*** and ***antisymmetric part*** are defined, respectively, by the classic expressions

$$M_{(ab)} = \tfrac{1}{2}(M_{ab} + M_{ba}), \qquad M_{[ab]} = \tfrac{1}{2}(M_{ab} - M_{ba}).$$

The operations of symmetrisation and antisymmetrisation can be extended in a natural way to higher rank tensors. In particular, it is noticed that for a rank 3 covariant tensor T_{abc} one has

$$T_{abc} = \tfrac{1}{3!}(T_{abc} + T_{bca} + T_{cab} - T_{acb} - T_{cba} - T_{bac}).$$

If a tensor $S_{a_1 \cdots a_l}$ is symmetric with respect to the indices $a_1, \ldots, a_l$ then one writes $S_{a_1\cdots a_l} = S_{(a_1\cdots a_l)}$. Similarly, if $A_{a_1\cdots a_l}$ is antisymmetric with respect to $a_1, \ldots, a_l$, one writes $A_{a_1\cdots a_l} = A_{[a_1\cdots a_l]}$.

Consistent with the abstract index notation for tensors, it is convenient to introduced a similar convention to denote the various tensor spaces. Accordingly the bundle $\mathfrak{T}^k_l(\mathcal{M})$ will, in the sequel denoted by $\mathfrak{T}^{a_1\cdots a_k}{}_{b_1\cdots b_l}(\mathcal{M})$. In particular, in this notation the tangent bundle $T(\mathcal{M})$ is denoted by $\mathfrak{T}^a(\mathcal{M})$ while the cotangent bundle $T^*(\mathcal{M})$ is given by $\mathfrak{T}_a(\mathcal{M})$.

A further discussion of the abstract index notation which specific remarks for the treatment of spinors is given in Section 3.1.4.

2.3 Maps between manifolds

The purpose of this section is to discuss certain features of maps between two manifolds $\mathcal{N}$ and $\mathcal{M}$. These manifolds can be the same.

2.3.1 Pull-backs and push-forwards

A map $\varphi : \mathcal{N} \to \mathcal{M}$ is said to be smooth (C^∞) if for every smooth function $f : \mathcal{M} \to \mathbb{R}$ (i.e. $f \in \mathfrak{X}(\mathcal{M})$), the composition $\varphi^* f \equiv f \circ \varphi : \mathcal{N} \to \mathbb{R}$ is also smooth . Given $p \in \mathcal{N}$, let $T_p(\mathcal{N})$, $T_{\varphi(p)}(\mathcal{M})$ denote, respectively, the tangent spaces at $p \in \mathcal{N}$ and $\varphi(p) \in \mathcal{M}$. The map $\varphi : \mathcal{N} \to \mathcal{M}$ induces a map $\varphi_* : T_p(\mathcal{N}) \to T_{\varphi(p)}(\mathcal{M})$, the ***push-forward*** via the formula

$$(\varphi_* \boldsymbol{v}) f(p) \equiv \boldsymbol{v}(f \circ \varphi)(p), \qquad \boldsymbol{v} \in T_p(\mathcal{N}).$$

It can be readily verified that φ_* so defined is a $\mathfrak{X}$-linear map —that is, given $\boldsymbol{v}$, $\boldsymbol{u} \in T(\mathcal{N})$ and a smooth function f one has $\varphi_*(f\boldsymbol{v} + \boldsymbol{u}) = f\varphi_*\boldsymbol{v} + \varphi_*\boldsymbol{u}$. Note that the above definition is done in a point-wise manner. *Smooth vector fields do not push-forward to smooth vector fields, except in the case of diffeomorphisms.* For example, if φ is not surjective (onto), then there is no way of deciding what vector to assign to a point which is not on the image of φ. If φ is not injective, then for some points of $\mathcal{M}$, there may be several different vectors obtained as push-forwards of a vector on $\mathcal{N}$. Given $\varphi : \mathcal{N} \to \mathcal{M}$ a diffeomorphism. For every $\boldsymbol{v} \in T(\mathcal{N})$ there exists a unique vector field on $T(\mathcal{M})$ obtained as the pull-back of $\boldsymbol{v}$ —see Lee (2002).

The push-forward $\varphi_* : T(\mathcal{N}) \to T(\mathcal{M})$ can be used, in turn, to define a map $\varphi^* : T^*(\mathcal{M}) \to T^*(\mathcal{N})$, the ***pull-back*** as

$$\langle \varphi^* \boldsymbol{\omega}, \boldsymbol{v} \rangle \equiv \langle \boldsymbol{\omega}, \varphi_* \boldsymbol{v} \rangle, \qquad \boldsymbol{\omega} \in T^*\mathcal{M}, \qquad \boldsymbol{v} \in T(\mathcal{N}).$$

Again, it can be readily verified that φ^* so defined is $\mathfrak{X}$-linear: $\varphi^*(f\boldsymbol{\omega} + \boldsymbol{\zeta}) = f^*\varphi^*\boldsymbol{\omega} + \varphi^*\boldsymbol{\zeta}$ for $\boldsymbol{\omega}$, $\boldsymbol{\zeta} \in T^*(\mathcal{M})$. The pull-back commutes with the differential $\mathbf{d}$ —that is, $\varphi^*(\mathbf{d}f) = \mathbf{d}(\varphi^* f)$. *Contrary to the case of psu-forwards, pull-backs of covector fields always lead to smooth co-vector fields. There is no ambiguity in the construction.* In the case that $\varphi : \mathcal{N} \to \mathcal{M}$ is a diffeomorphism, then the inverse pull-back $(\varphi^*)^{-1}$ is well defined so that covectors can be pull-backed from $T^*(\mathcal{N})$ to $T^*(\mathcal{M})$.

The operations of push-forward and pull-back can be extended in the natural way, respectively, to arbitrary contravariant and covariant tensors. The case of most relevance for the subsequent discussion is that of a covariant tensor of rank 2, $\boldsymbol{g} \in \mathfrak{T}_2(\mathcal{M})$. Its pull-back $\varphi^*\boldsymbol{g} \in \mathfrak{T}_2(\mathcal{N})$ satisfies

$$(\varphi^*\boldsymbol{g})(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}(\varphi_*\boldsymbol{u}, \varphi_*\boldsymbol{v}), \qquad \boldsymbol{u},\, \boldsymbol{v} \in T(\mathcal{N}).$$

2.3.2 Lie derivatives

The particular case of smooth maps of the manifold to itself, $\varphi : \mathcal{M} \to \mathcal{M}$, leads in a natural way to the notion of Lie derivative. Given a vector $\boldsymbol{v}$, the ***Lie derivative*** $\pounds_{\boldsymbol{v}}$ measures the change of a tensor field along the integral curves of $\boldsymbol{v}$.

In what follows, let $f \in \mathfrak{X}(\mathcal{M})$ denote a smooth function and $\boldsymbol{u}$, $\boldsymbol{v} \in T(\mathcal{M})$, $\boldsymbol{\alpha} \in T^*(\mathcal{M})$. The action of $\pounds_{\boldsymbol{v}}$ on functions, vectors and is given by

$$\pounds_{\boldsymbol{v}} f = \boldsymbol{v} f,$$
$$\pounds_{\boldsymbol{v}} \boldsymbol{u} = [\boldsymbol{v}, \boldsymbol{u}],$$

The Lie derivative can be extended to act on covectors by requiring that

$$\pounds_{\boldsymbol{v}}\langle \boldsymbol{\alpha}, \boldsymbol{u}\rangle = \langle \pounds_{\boldsymbol{v}}\boldsymbol{\alpha}, \boldsymbol{u}\rangle + \langle \boldsymbol{\alpha}, \pounds_{\boldsymbol{v}}\boldsymbol{u}\rangle.$$

A coordinate expression can be obtained from the latter. The action of $\pounds_{\boldsymbol{v}}$ can be extended to arbitrary tensor fields by means of the Leibnitz rule

$$\pounds_{\boldsymbol{v}}(\boldsymbol{S} \otimes \boldsymbol{T}) = \pounds_{\boldsymbol{v}}\boldsymbol{S} \otimes \boldsymbol{T} + \boldsymbol{S} \otimes \pounds_{\boldsymbol{v}}\boldsymbol{T}.$$

The reader interested in the derivation of the above expressions and their precise relation to the notions of push-forward and pull-back of tensor fields is refered to e.g. Stewart (1991) where a list of coordinate expressions for the computation of the derivatives is also provided.

2.4 Connections, torsion and curvature

The purpose of the present section is to discuss in some extent the properties of linear connections and the associated notions of covariant derivatives, torsion and curvature.

2.4.1 Covariant derivatives and connections

The notion of linear connection allows to relate the tensors at different point of the manifold $\mathcal{M}$.

Definition *A* ***linear connection****, or* ***connection*** *for short, is a map* $\boldsymbol{\nabla} : \mathfrak{T}^1(\mathcal{M}) \times \mathfrak{T}^1(\mathcal{M}) \rightarrow \mathfrak{T}^1(\mathcal{M})$ *sending the pair of vector fields* $(\boldsymbol{u}, \boldsymbol{v})$ *to a vector field* $\nabla_{\boldsymbol{v}}\boldsymbol{u}$ *satisfying:*

(a) $\nabla_{\boldsymbol{u}+\boldsymbol{v}}\boldsymbol{w} = \nabla_{\boldsymbol{u}}\boldsymbol{w} + \nabla_{\boldsymbol{v}}\boldsymbol{w}$*;*
(b) $\nabla_{\boldsymbol{u}}(\boldsymbol{v} + \boldsymbol{w}) = \nabla_{\boldsymbol{u}}\boldsymbol{v} + \nabla_{\boldsymbol{u}}\boldsymbol{w}$*;*
(c) $\nabla_{f\boldsymbol{u}}\boldsymbol{v} = f\nabla_{\boldsymbol{u}}\boldsymbol{v}$*;*
(d) $\nabla_{\boldsymbol{u}}(f\boldsymbol{v}) = \boldsymbol{v}(f)\boldsymbol{v} + f\nabla_{\boldsymbol{u}}\boldsymbol{v}$,

for $f \in \mathfrak{X}(\mathcal{M})$*. The vector* $\nabla_{\boldsymbol{u}}\boldsymbol{v}$ *is called the* ***covariant derivative of v with respect to u****.*

Any manifold admits a connection. In 4 dimensions this is done by the specification of 4^3 functions on the spacetime manifold $\mathcal{M}$ —see e.g. Willmore (1993). The reason behind this result become more transparent one the so-called *connection coefficients* have been introduced —see Section 2.6.

As a consequence of the *Leibnitz rule* in *(d)* $\nabla_{\boldsymbol{u}}\boldsymbol{v}$ is not $\mathfrak{X}$-linear in $\boldsymbol{u}$. However, it is linear in $\boldsymbol{u}$. Thus, using Lemma 1 for fixed second argument it defines a mixed $(1,1)$-tensor. Using abstract index notation the latter is denoted by $\nabla_a v^b$ —so that $\nabla_a v^b \in \mathfrak{T}_a{}^b(\mathcal{M})$.

From the discussion in the previous paragraph it follows that in practice one can regard the connection $\boldsymbol{\nabla}$ as a map $\nabla_a : \mathfrak{T}^b(\mathcal{M}) \rightarrow \mathfrak{T}_a{}^b(\mathcal{M})$. Moreover, a connection $\boldsymbol{\nabla}$ induces a map $\nabla_a : \mathfrak{T}_b(\mathcal{M}) \rightarrow \mathfrak{T}_{ab}(\mathcal{M})$ via

$$(\nabla_a \omega_b) v^b = \nabla_a(\omega_b v^b) - \omega_b(\nabla_b v^b).$$

This map is fixed if one requires the *Leibnitz rule* to hold between the product of a vector a a 1-form. In order to extend the covariant derivative to arbitrary tensors one uses again the Leibnitz rule. For example, from:

$$\begin{aligned}\nabla_e(\omega_a T^a{}_{bcd} u^b v^c w^d) = {} & \omega_a(\nabla_e T^a{}_{bcd}) u^b v^c w^d + (\nabla_e \omega_a) T^a{}_{bcd} u^b v^c w^d \\ & + \omega_a T^a{}_{bcd}(\nabla_e u^b) v^c w^d + \omega_a T^a{}_{bcd} u^b (\nabla_e v^c) w^d \\ & + \omega_a T^a{}_{bcd} u^b v^c (\nabla_e w^d),\end{aligned}$$

it follows that

$$\begin{aligned}(\nabla_e T^a{}_{bcd}) \omega_a u^b v^c w^d = {} & \nabla_e(\omega_a T^a{}_{bcd} u^b v^c w^d) - (\nabla_e \omega_a) T^a{}_{bcd} u^b v^c w^d \\ & - \omega_a T^a{}_{bcd}(\nabla_e u^b) v^c w^d - \omega_a T^a{}_{bcd} u^b (\nabla_e v^c) w^d \\ & - \omega_a T^a{}_{bcd} u^b v^c (\nabla_e w^d).\end{aligned}$$

so that one obtains a linear map $\mathfrak{T}^a{}_{bcd}(\mathcal{M}) \to \mathfrak{T}_e{}^a{}_{bcd}(\mathcal{M})$.

The subsequent discussion will make use of the ***commutator of covariant derivatives***. This is defined as

$$[\nabla_a, \nabla_b] \equiv 2\nabla_{[a}\nabla_{b]}.$$

One has that

$$[\nabla_a, \nabla_b](T_{\mathcal{A}} + S_{\mathcal{A}}) = [\nabla_a, \nabla_b]T_{\mathcal{A}} + [\nabla_a, \nabla_b]S_{\mathcal{A}},$$
$$[\nabla_a, \nabla_b](S_{\mathcal{A}}R_{\mathcal{B}}) = ([\nabla_a, \nabla_b]T_{\mathcal{A}})R_{\mathcal{B}} + T_{\mathcal{A}}([\nabla_a, \nabla_b]R_{\mathcal{B}},$$

where $_{\mathcal{A}}$ and $_{\mathcal{B}}$ denote an arbitrary string of (covariant and contravariant) indices.

Covariant derivatives and derivations (cf. Definition 2.2.1) on a manfiold are related in a natural way: given a derivation $\mathcal{D}$ and a connection $\boldsymbol{\nabla}$ on $\mathcal{M}$ there exists a unique $\boldsymbol{v} \in T(\mathcal{M})$ such that $\mathcal{D}f = v^a\nabla_a f$ for any $f \in \mathfrak{X}(\mathcal{M})$ —see e.g. O'Neill (1983).

2.4.2 Torsion of a connection

The notion of ***torsion*** arises naturally from the analysis of the effect of the commutator of covariant derivatives on scalar fields. For covenience the abstract index notation is used. To this end consider $x^{ab} \in \mathfrak{T}^{ab}(\mathcal{M})$ and $f,\ g \in \mathfrak{X}(\mathcal{M})$. One readily has that

$$x^{ab}[\nabla_a, \nabla_b](f + g) = x^{ab}[\nabla_a, \nabla_b]f + x^{ab}[\nabla_a, \nabla_b]g,$$
$$x^{ab}[\nabla_a, \nabla_b](fg) = (x^{ab}[\nabla_a, \nabla_b]f)g + f(x^{ab}[\nabla_a, \nabla_b]g).$$

It follows from the later that the operator $x^{ab}[\nabla_a, \nabla_b]$ must be a derivation —see Definition 2.2.1. Thus, there exists $u^a \in \mathfrak{T}^a(\mathcal{M})$ such that

$$x^{ab}[\nabla_a, \nabla_b] = u^a\nabla_a. \tag{2.5}$$

The map $x^{ab} \mapsto u^a\nabla_a$ defined by (2.5) is $\mathfrak{X}$-linear. It defines a tensor $\boldsymbol{\Sigma}$, the ***torsion tensor*** of the connection $\boldsymbol{\nabla}$, via $u^c = x^{ab}\Sigma_a{}^c{}_b$. Hence,

$$\nabla_a\nabla_b f - \nabla_b\nabla_a f = \Sigma_a{}^c{}_b\nabla_c f, \qquad f \in \mathfrak{X}(\mathcal{M}). \tag{2.6}$$

One readily sees that

$$\Sigma_a{}^c{}_b = -\Sigma_b{}^c{}_a.$$

That is, the torsion is an antisymmetric tensor. If a connection $\boldsymbol{\nabla}$ is such that $\Sigma_a{}^c{}_b = 0$, then it is said to be ***torsion-free***.

Remark. Alternatively, one could have defined the torsion via the relation

$$\boldsymbol{\Sigma}(\boldsymbol{u},\boldsymbol{v}) = \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u},\boldsymbol{v}], \qquad \boldsymbol{u},\,\boldsymbol{v}\in T(\mathcal{M}). \tag{2.7}$$

From this definition it can the be verified that $\boldsymbol{\Sigma}$ as given by the above definition is $(1,2)$-tensor. The discussion in this section can then, be seen to follow from (2.7). Hence $\boldsymbol{\Sigma}$ is the abstract index counterpart of $\Sigma_a{}^c{}_b$.

2.4.3 Curvature of a connection

In order to discuss the notion of curvature of a connection it is convenient to define the following *modified commutator* of covariant derivatives

$$[\![\nabla_a,\nabla_b]\!] \equiv [\nabla_a,\nabla_b] - \Sigma_a{}^c{}_b\nabla_c.$$

Clearly, one has that $[\![\nabla_a,\nabla_b]\!]f = 0$ for $f\in\mathfrak{X}(\mathcal{M})$. Thus,

$$[\![\nabla_a,\nabla_b]\!](fT_{\mathcal{A}}) = f[\![\nabla_a,\nabla_b]\!]T_{\mathcal{A}}$$

for ${}_{\mathcal{A}}$ denoting, again, an abitrary string of covariant or contravariant indices. In particular, one has that

$$[\![\nabla_a,\nabla_b]\!](fu^c) = f[\![\nabla_a,\nabla_b]\!]u^c,$$
$$[\![\nabla_a,\nabla_b]\!](u^c+v^c) = [\![\nabla_a,\nabla_b]\!]u^c + [\![\nabla_a,\nabla_b]\!]v^c.$$

From the previous expressions one concludes that the map $u^d\mapsto[\![\nabla_a,\nabla_b]\!]u^d$ is $\mathfrak{X}$-linear. Thus, using Lemma 1 it defines a tensor field $R^d{}_{cab}$, the ***Riemann curvature tensor*** of the connection $\boldsymbol{\nabla}$. One can write

$$[\![\nabla_a,\nabla_b]\!]u^d = \left([\nabla_a,\nabla_b] - \Sigma_a{}^c{}_b\nabla_c\right)u^d = R^d{}_{cab}u^c. \tag{2.8}$$

Alternatively, one has that

$$\left(\nabla_a\nabla_b - \nabla_b\nabla_a\right)u^d = R^d{}_{cab}u^c + \Sigma_a{}^c{}_b\nabla_c u^d.$$

The antisymmetry of $[\![\nabla_a,\nabla_b]\!]$ on the indices ${}_a$ and ${}_b$ is readily inherited by the Rieman curvature tensor, so that

$$R^d{}_{cab} = -R^d{}_{cba}.$$

The action of the commutator of covariant derivatives can be extended to other tensors using the Leibnitz rule. For example, from

$$[\![\nabla_a,\nabla_b]\!](\omega_d v^d) = \left([\![\nabla_a,\nabla_b]\!]\omega_d\right)v^d + \omega_d[\![\nabla_a,\nabla_b]\!]v^d,$$

one can conclude that

$$\left(\nabla_a\nabla_b - \nabla_b\nabla_a\right)\omega_d = -R^c{}_{dab}\omega_c + \Sigma_a{}^c{}_b\nabla_c\omega_d.$$

Similarly, evaluating $[\![\nabla_a, \nabla_b]\!](S^d{}_{ef}\omega_d u^e v^f)$ one concludes that

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\, S^d{}_{ef} = R^d{}_{cab}S^c{}_{ef} - R^c{}_{eab}S^d{}_{cf} - R^c{}_{eab}S^d{}_{ec} + \Sigma_a{}^c{}_b\nabla_c S^d{}_{ef}.$$

Remark. As in the case of the torsion, the curvature can be defined in an alternative way via the relation

$$\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}, \qquad \boldsymbol{u},\, \boldsymbol{v},\, \boldsymbol{w} \in T(\mathcal{M}), \quad (2.9)$$

where the expression $\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}$ corresponds to $R^d{}_{cab}w^c u^a v^b$ in abstract index notation. The discussion of the present section, and in particular expression (2.8), can be deduced from (2.9). The presentation followed leads to a more direct extension to spinorial structurs —see Chapter 3.

Bianchi identities

With the aim of investigating further symmetries of the curvature tensor, consider the triple derivative $\nabla_{[a}\nabla_b\nabla_{c]}f$ of $f \in \mathfrak{X}(\mathcal{M})$. A computation shows, on the one hand, that

$$\begin{aligned} 2\nabla_{[a}\nabla_b\nabla_{c]}f &= 2\nabla_{[[a}\nabla_{b]}\nabla_{c]}f = [\![\nabla_{[a}, \nabla_b]\!]\nabla_{c]}f \\ &= \Sigma_{[a}{}^d{}_b\nabla_{|d|}\nabla_{c]}f - R^d{}_{[cab]}\nabla_d f, \end{aligned}$$

and on the other that

$$\begin{aligned} 2\nabla_{[a}\nabla_b\nabla_{c]}f &= 2\nabla_{[a}\nabla_{[b}\nabla_{c]]}f \\ &= \nabla_{[a}[\nabla_b, \nabla_{c]}]f = \nabla_{[a}\left(\Sigma_b{}^d{}_{c]}\nabla_d f\right), \\ &= \nabla_{[a}\Sigma_b{}^d\nabla_{d]}f + \Sigma_{[b}{}^d{}_c\nabla_{a]}\nabla_d f. \end{aligned}$$

Putting these two computations together and using the definition of the torsion tensor, equation (2.6), one concludes that

$$\nabla_{[a}\Sigma_b{}^d{}_{c]}\nabla_d f + R^d{}_{[cab]}\nabla_d f + \Sigma_{[a}{}^d{}_b\Sigma_{c]}{}^e{}_d\nabla_e f = 0.$$

As the scalar field f is arbitrary, one concludes then that

$$\nabla_{[a}\Sigma_b{}^d{}_{c]} + R^d{}_{[cab]} + \Sigma_{[a}{}^d{}_b\Sigma_{c]}{}^e{}_d = 0. \quad (2.10)$$

This is the so-called ***first Bianchi identity***. In the case of a *torsion-free connection* it takes the more familiar form

$$R^d{}_{[cab]} = 0.$$

Notice that because of the antisymmetry in the last two indices, the latter can be written as

$$R^d{}_{cab} + R^d{}_{abc} + R^d{}_{bca} = 0.$$

Next, consider the action of $\nabla_{[a}\nabla_b\nabla_{c]}$ on a vector field v^d. As in the case of the first Bianchi identity, one can compute this object in two different ways. On the one hand one has that

$$\begin{aligned}2\nabla_{[a}\nabla_b\nabla_{c]}v^d &= 2\nabla_{[[a}\nabla_{b]}\nabla_{c]}v^d \\ &= [\nabla_{[a},\nabla_b]\nabla_{c]}v^d \\ &= [\![\nabla_{[a},\nabla_a]\!]\nabla_{c]}v^d + \Sigma_{[a}{}^e{}_b\nabla_{|e|}\nabla_{c]}v^d \\ &= -R^e{}_{[cab]}\nabla_e v^d + R^d{}_{e[ab}\nabla_{c]}v^e + \Sigma_{[a}{}^e{}_b\nabla_{|e|}\nabla_{c]}v^d,\end{aligned}$$

and on the other that

$$\begin{aligned}2\nabla_{[a}\nabla_b\nabla_{c]}v^d &= 2\nabla_{[a}\nabla_{[b}\nabla_{c]]}v^d \\ &= 2\nabla_{[a}[\![\nabla_b,\nabla_{c]}]\!]v^d + \nabla_{[a}\left(\Sigma_b{}^e{}_{c]}\nabla_e v^d\right), \\ &= \nabla_{[a}R^d{}_{|e|bc}v^e + R^d{}_{e[bc}\nabla_{a]}v^e + \nabla_{[a}\Sigma_b{}^e{}_{c]}\nabla_e v^d + \Sigma_{[b}{}^e{}_c\nabla_{a]}\nabla_e v^d.\end{aligned}$$

Equating the two expressions and using the first Bianchi identity, equation (2.10), to eliminate covariant derivatives of the torsion tensor one concludes that

$$\nabla_{[a}R^d{}_{|e|bc]} + \Sigma_{[a}{}^f{}_bR^d{}_{|e|c]f} = 0.$$

This is the so-called ***second Bianchi identity***. For a *torsion-free connection* one obtains the well known expression

$$\nabla_{[a}R^d{}_{|e|bc]} = 0. \tag{2.11}$$

2.4.4 Change of connection

Consider two connections $\boldsymbol{\nabla}$ and $\bar{\boldsymbol{\nabla}}$ on the manifold $\mathcal{M}$. A natural question to ask is whether there is any relation between these connections and their associated torsion and curvature tensors. By definition one has that

$$(\bar{\nabla}_a - \nabla_a)f = 0, \qquad f \in \mathfrak{X}(\mathcal{M}).$$

However, one also has that

$$(\bar{\nabla}_a - \nabla_a)(fv^a) = f(\bar{\nabla}_a - \nabla_a)v^a.$$

From here it follows that the map $v^b \mapsto (\bar{\nabla}_a - \nabla_a)v^a$ is $\mathfrak{X}$-linear, so that invoquing Lemma 1 there exists a tensor field, the ***transition tensor*** $Q_a{}^b{}_c$, such that

$$(\bar{\nabla}_a - \nabla_a)v^b = Q_a{}^b{}_cv^c. \tag{2.12}$$

Now, from

$$(\bar{\nabla}_a - \nabla_a)(\omega_b v^b) = 0,$$

one readily concludes that

$$(\bar{\nabla}_a - \nabla_a)\omega_b = -Q_a{}^c{}_b \omega_c. \tag{2.13}$$

A different choice of covariant derivatives gives rise to a different choice of transition tensor. The set of connections over a manifold $\mathcal{M}$ is an affine space: given a connection ∇ on the manifold, any other connection can be achieved by a suitable choice of transition tensor. If $\boldsymbol{Q}$ denotes the index-free version of the tensor $Q_a{}^b{}_c$, then the relation between the connection $\boldsymbol{\nabla}$ and $\bar{\boldsymbol{\nabla}}$ will be denoted, in a schematic way, as

$$\bar{\boldsymbol{\nabla}} - \boldsymbol{\nabla} = \boldsymbol{Q}.$$

In Chapter (5) specific forms for the transition tensor will be investigated.

Transformation of the torsion and the curvature

A direct computation using equations (2.6) and (2.12) renders the following relation between the torsion tensors of the connections $\bar{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$:

$$\bar{\Sigma}_a{}^c{}_b - \Sigma_a{}^c{}_b = -2Q_{[a}{}^c{}_{b]}. \tag{2.14}$$

In particular, it follows that if $Q_a{}^c{}_b = \frac{1}{2}\Sigma_a{}^c{}_b$, then $\bar{\Sigma}_a{}^c{}_a = 0$. That is, *it is always possible to construct a connection which is torsion free.*

An analogous, albeit lengthier computation using equations (2.6) and (2.8) renders the following relation between the respective curvature tensors:

$$\bar{R}^c{}_{dab} - R^c{}_{dab} = 2\nabla_{[a}Q_{b]}{}^c{}_d - \Sigma_a{}^e{}_b Q_e{}^c{}_d + 2Q_{[a}{}^c{}_{|e|}Q_{b]}{}^e{}_d. \tag{2.15}$$

2.4.5 The geodesic and geodesic deviation equations

Given a covariant derivative $\boldsymbol{\nabla}$, one can introduce the notion of ***parallel propagation***. Given $\boldsymbol{u}$, $\boldsymbol{v} \in T(\mathcal{M})$, then $\boldsymbol{u}$ is said to be parallely propagated in the direction of $\boldsymbol{v}$ if it satisfies the parallel propagation equation $\nabla_{\boldsymbol{v}}\boldsymbol{u} = 0$.

A ***geodesic*** $\gamma \subset \mathcal{M}$ is a curve whose tangent vector is parallely propagated along itself. Following the convention of Section 2.2.1, let $\dot{\boldsymbol{x}}$ denote the tanget vector to γ. One has that

$$\nabla_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = 0. \tag{2.16}$$

A ***congruence of geodesics*** is the set of integral curves of a (tangent) vector field $\dot{\boldsymbol{x}}$ satisfying equation (2.16). Any vector $\boldsymbol{z}$ such that $[\dot{\boldsymbol{x}}, \boldsymbol{z}] = 0$ is called a ***deviation vector*** of the congruence of geodesics. *Assuming that* *the connection* $\boldsymbol{\nabla}$ *is torsion-free* so that $\nabla_{\dot{\boldsymbol{x}}}\boldsymbol{z} = \nabla_{\boldsymbol{z}}\dot{\boldsymbol{x}}$, a computation shows that $\boldsymbol{z}$ satisfies the ***geodesic deviation equation***

$$\nabla_{\dot{\boldsymbol{x}}}\nabla_{\dot{\boldsymbol{x}}}\boldsymbol{z} = \boldsymbol{R}(\dot{\boldsymbol{x}}, \boldsymbol{z})\dot{\boldsymbol{x}}.$$

2.5 Metric tensors

A ***metric*** on the manifold $\mathcal{M}$ is a symmetric rank 2 covariant tensor field $\boldsymbol{g}$ —in abstract index notation to be denoted by g_{ab}. The metric tensor $\boldsymbol{g}$ is said to be ***non-degenerate*** if $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = 0$ for all $\boldsymbol{u}$ if and only if $\boldsymbol{v} = 0$. In the sequel, and unless otherwise explicitly stated, it is assumed that all the metrics under consideration are non-degerate. If $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = 0$, then the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be ***orthogonal***. Pointwise, the components $g_{\boldsymbol{ab}} \equiv g(\boldsymbol{e_a}, \boldsymbol{e_b})$ with respect to a basis $\{\boldsymbol{e_a}\}$ define a symmetric $n \times n$ matrix. Because of the symmetry, the matrix $(g_{\boldsymbol{ab}})$ has n real eigenvalues. The ***signature*** of $\boldsymbol{g}$ is the difference between the number of positive and negative eigenvalues. If the signature is n or $-n$, then $\boldsymbol{g}$ is said to be a ***Riemannian metric***. If the signature $\pm(n-2)$, then $\boldsymbol{g}$ is a ***Lorentzian metric***.

From the non-degeneracy of $\boldsymbol{g}$ it follows that there exists a unique contravariant rank 2 tensor to be denoted, depending on the context, by $\boldsymbol{g}^\sharp$ or g^{ab} such that $g_{ab}g^{bc} = \delta_a{}^c$. In terms of components with respect to a basis this menas that the matrices $(g_{\boldsymbol{ab}})$ and $(g^{\boldsymbol{ab}})$ are inverses of each other. Accordingly, $\boldsymbol{g}^\sharp$ is also non-degenerate and one obtains an isomorphism between the vector spaces $T_p\mathcal{M}$ and $T^*_p\mathcal{M}$. More precisely, given $\boldsymbol{v} \in T_p\mathcal{M}$ then $\boldsymbol{v}^\flat \equiv \boldsymbol{g}(\boldsymbol{v}, \cdot) \in T^*_p\mathcal{M}$ as $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}$ for any $\boldsymbol{u} \in T_p\mathcal{M}$. In what follows $\boldsymbol{g}(\boldsymbol{v}, \cdot)$ will be denoted by $\boldsymbol{v}^\flat$. Similarly, given $\omega \in T^*_p\mathcal{M}$ one has that $\omega^\sharp \equiv \boldsymbol{g}^\sharp(\omega, \cdot) \in T_p\mathcal{M}$. In terms of abstract indices, the operations $^\flat$ (flat) and $^\sharp$ (sharp) correspond to the usual operations of ***lowering and raising of indices*** by means of g_{ab} and g^{ab}:

$$v_a \equiv g_{ab}v^b, \qquad \omega^a \equiv g^{ab}\omega_b.$$

Clearly, the operations $^\flat$ and $^\sharp$ are inverses of each other. They can be extended in a natural way to tensors of arbitrary rank.

Given two manifolds $\mathcal{M}$ and $\bar{\mathcal{M}}$ with metrics given by, respectively, $\boldsymbol{g}$ and $\bar{\boldsymbol{g}}$, a diffeomorphism $\varphi : \mathcal{M} \to \bar{\mathcal{M}}$ is called an ***isometry*** if $\varphi^*\bar{\boldsymbol{g}} = \boldsymbol{g}$. If an isometry exists then the pairs $(\mathcal{M}, \boldsymbol{g})$ and $(\bar{\mathcal{M}}, \bar{\boldsymbol{g}})$ are said to be ***isometric***. If $\mathcal{M} = \bar{\mathcal{M}}$ and $\boldsymbol{g} = \bar{\boldsymbol{g}}$, one speaks of an isometry of $\mathcal{M}$.

Remark. In this book the negative sign in the definition of signature will be taken as conventional. Most Lorentzian metrics to be considered will be associated to 4-dimensional manifolds. Thus, unless explicitly stated, Lorentzian metrics are assumed to have signature -2. This convention leads, in a natural way, to consider 3-dimensional ***negative-definite*** Riemannian metrics —that is, metrics with signature -3. Only 3-dimensional manifolds will be considered. In the sequel, the symbol $\boldsymbol{g}$ will be used to denote a generic Lorentzian metric while $\boldsymbol{h}$ will be used for a generic negative-definite Riemaniann metric.

Specifics for Lorentzian metrics

Following the terminology of General Relativity a pair $(\mathcal{M}, \boldsymbol{g})$ consisting of a 4-dimensional manifold and a Lorentzian metric will be called a ***space-time***. The metric $\boldsymbol{g}$ can be used to classify vectors in a pointwise manner as ***timelike***, ***null*** or ***spacelike*** depending on whether $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) > 0$, $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$ or $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) < 0$, respectively. A basis $\{\boldsymbol{e_a}\}$ is said to be ***orthonormal*** if

$$\boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e}_b) = \eta_{\boldsymbol{ab}}, \qquad \eta_{\boldsymbol{ab}} \equiv \mathrm{diag}(1, -1, -1, -1).$$

It follows that $\boldsymbol{g}$ can be written as

$$\boldsymbol{g} = \eta_{\boldsymbol{ab}} \boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}},$$

where $\{\boldsymbol{\omega}^a\}$ denotes the co-frame dual to $\{\boldsymbol{e}_a\}$.

The set of null vectors at a point $p \in \mathcal{M}$ is called the ***null cone at*** p and will be denoted by $\mathcal{C}_p$. By definition timelike vectors lie inside the light cone while spacelike ones lie outside. The null cone is made of two-half cones. If one of these half cones can be singled out and called the ***future half cone*** $\mathcal{C}_p^+$ and the other the ***past half cone*** $\mathcal{C}_p^-$ then $T_p\mathcal{M}$ is said to be ***time oriented***. A timelike vector inside $\mathcal{C}_p^+$ is said to be ***future directed***; similarly a timelike vector inside $\mathcal{C}_p^-$ is called the ***past directed***. If $T\mathcal{M}$ can be time oriented in a continuous manner for all $p \in \mathcal{M}$, then $(\mathcal{M}, \boldsymbol{g})$ is said to be a ***time oriented spacetime***. A curve $\gamma \subset \mathcal{M}$ with *timelike, future oriented* tangent vector $\dot{\boldsymbol{x}}$ is said to be ***parametrised by its proper time*** if $\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1$.

Specifics for Riemannian metrics

A Riemannian metric $\boldsymbol{h}$ endows the tangent spaces of the manifold with an inner product. Because of the signature conventions being used, this inner product will, naturally, be negative definite. An important basic result of Riemannian geometry is that every differential manifold admits a Riemannian metric. The proof of this argument heavily relies on the paracompactness of the manifold —see e.g. Choquet-Bruhat et al. (1982).

In the case of a Riemannian metric $\boldsymbol{h}$, a basis $\{\boldsymbol{e}_{\boldsymbol{\alpha}}\}$ is said to be ***orthonormal*** if

$$\boldsymbol{h}(\boldsymbol{e}_{\boldsymbol{\alpha}}, \boldsymbol{e}_{\boldsymbol{\beta}}) = -\delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}, \qquad \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} \equiv \mathrm{diag}(1,1,1).$$

Thus, using the associated coframe basis $\{\boldsymbol{\omega}^{\boldsymbol{\alpha}}\}$ one can write

$$\boldsymbol{h} = -\delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}\boldsymbol{\omega}^{\boldsymbol{\alpha}} \otimes \boldsymbol{\omega}^{\boldsymbol{\beta}}.$$

2.5.1 Metric connections and Levi-Civita connections

Two conditions which are usually required from a connection are metric compatibility and torsion-freeness. In this section the (well-known) consequences of these assumptions are briefly reviewed.

Metric connections

A connection $\boldsymbol{\nabla}$ on $\mathcal{M}$ is said to be ***metric with respect to*** $\boldsymbol{g}$ if $\boldsymbol{\nabla}\boldsymbol{g} = 0$ (i.e. $\nabla_a g_{bc} = 0$). The Riemman curvature tensor of the connection ∇ acquires, by virtue of the metricity condition a further symmetry. This can be better seen by applying the modified commutator $[\![\nabla_a, \nabla_b]\!]$ to the metric g_{ab}. On the one hand, by the assumption of metricity one readily has that $[\![\nabla_a, \nabla_b]\!] g_{cd} = 0$, while on the other that

$$[\![\nabla_a, \nabla_b]\!] g_{cd} = -R^e{}_{cab} g_{ed} - R^e{}_{dab} g_{ce} = -R_{dcab} - R_{cdab},$$

where $R_{dcab} \equiv R^e{}_{cab}$. Hence, one concludes that

$$R_{cdab} = -R_{dcab}. \tag{2.17}$$

The Levi-Civita connection

A connection $\boldsymbol{\nabla}$ is said to be the ***Levi-Civita connection of the metric*** $\boldsymbol{g}$ if ∇ is *torsion-free* and *metric* with respect to $\boldsymbol{g}$. The *Fundamental Theorem of Riemannian Geometry* (also valid in the Lorentzian case) ensures that the Levi-Civita connection of a metric $\boldsymbol{g}$ is unique. The proof of this result is well-known and readily available in most books on Riemannian geometry —see e.g. Choquet-Bruhat et al. (1982). The Levi-Civita connection $\boldsymbol{\nabla}$ of the metric $\boldsymbol{g}$ is characterised by the so-called ***Koszul formula***

$$\begin{aligned} 2\boldsymbol{g}(\nabla_{\boldsymbol{v}}\boldsymbol{u}, \boldsymbol{w}) = {} & \boldsymbol{v}(\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{w})) + \boldsymbol{u}(\boldsymbol{g}(\boldsymbol{w}, \boldsymbol{v})) - \boldsymbol{w}(\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u})) \\ & -\boldsymbol{g}(\boldsymbol{v}, [\boldsymbol{u}, \boldsymbol{w}]) + \boldsymbol{g}(\boldsymbol{u}, [\boldsymbol{w}, \boldsymbol{v}]) + \boldsymbol{g}(\boldsymbol{w}, [\boldsymbol{v}, \boldsymbol{u}]). \end{aligned} \tag{2.18}$$

Of particular interest are the further symmetries that the Riemann tensor of a Levi-Civita connection posseses. First of all, because of the metricity, the curvature tensor has the symmetry of equation (2.17). Furthermore, as the connection is torsion free, the first Bianchi identity implies $R_{c[dab]} = 0$. From the above one readily has that

$$\begin{aligned} 2R_{cdab} &= R_{cdab} + R_{dcba} \\ &= -R_{cabd} - R_{cbda} - R_{dbac} - R_{dacb} \\ &= -R_{acdb} - R_{bcad} - R_{bdca} - R_{adbc} \\ &= R_{abcd} + R_{badc}. \end{aligned}$$

Hence, one recovers the well-known symmetry of interchange of pairs

$$R_{cdab} = R_{abcd}.$$

Characterisation of flatness

An open subset $\mathcal{U} \subset \mathcal{M}$ of a spacetime $(\mathcal{M}, \boldsymbol{g})$ is said to be flat if the metric $\boldsymbol{g}$ is isometric to the ***Minkowski metric***

$$\boldsymbol{\eta} = \eta_{\mu\nu}\mathbf{d}x^{\mu} \otimes \mathbf{d}x^{\nu}, \qquad \eta_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1).$$

In the case of a 3-dimensional Riemannian manifold $(\mathcal{S}, \boldsymbol{h})$, flatness corresponds to isometry with the 3-dimensional (negative definite) ***Euclidean metric***

$$\boldsymbol{\delta} = \delta_{\mu\nu}\mathbf{d}x^{\mu} \otimes \mathbf{d}x^{\nu}, \qquad \delta_{\mu\nu} \equiv \mathrm{diag}(1, 1, 1).$$

The Riemann tensor of a Levi-Civita connection provides a local characterisation of the flatness of a manifold. More precisely, the metric $\boldsymbol{g}$ (respectively $\boldsymbol{h}$) is flat on $\mathcal{U}$ if and only if $\boldsymbol{R} = 0$ on $\mathcal{U}$. The *if* part of the result follows by direct evaluation of the Riemann of the metric $\boldsymbol{\eta}$ (respectively $\boldsymbol{\delta}$). The *only if* part is much more challenging —see e.g. Choquet-Bruhat et al. (1982), page 310 for a proof.

Traces

A metric $\boldsymbol{g}$ on a manifold $\mathcal{M}$ allows to introduce a further operation between tensors which reduces the rank by 2: the ***trace*** with respect to $\boldsymbol{g}$. Given $\boldsymbol{T} \in \mathfrak{T}_2(\mathcal{M})$, its trace $\mathbf{trace}_{\boldsymbol{g}}\boldsymbol{T}$, is the scalar described in abstract index notation by $g^{ab}T_{ab}$. Noticing that $g^{ab}T_{ab} = T^a{}_a$, one sees that taking the trace of a tensor is a natural generalisation of the operation of contraction. The trace operation can be generalised in a natural way to an any pair of indices of the same type in an arbitrary tensor —for example, $g^{ac}M_{abcd}$ and g^{bc} denote the traces of M_{abcd} with respect to the first and third arguments and the second an third ones, respectively.

Given a symmetric tensor $T_{ab} = T_{(ab)} \in \mathfrak{T}_{ab}(\mathcal{M})$, its ***trace-free part*** $T_{\{ab\}}$ is given by

$$T_{\{ab\}} \equiv T_{ab} - \tfrac{1}{4} g_{ab} g^{cd} T_{cd}.$$

In the case of a 3-dimensional manifold $\mathcal{S}$ with metric $\boldsymbol{h}$, the above definition has to be modified to

$$T_{\{\alpha\beta\}} \equiv T_{\alpha\beta} - \tfrac{1}{3} h_{\alpha\beta} h^{\gamma\delta} T_{\gamma\delta},$$

for a symmetric tensor $T_{\alpha\beta} \in \mathfrak{T}_{\alpha\beta}(\mathcal{S})$. The operation of taking the trace-free part of a tensor can be extended to tensors of arbitrary rank. Unfortunately, the expressions to compute them become increasingly cumbersome. A more efficient approach to describe this operation is in terms of spinors —see Chapters 3 and 4. A tensor $M_{a_1 \cdots a_k}$ is said to be ***trace-free*** if $M_{a_1 \cdots a_k} = M_{\{a_1 \cdots a_k\}}$.

2.5.2 Decomposition of the Riemann tensor

In what follows, consider a spacetime $(\mathcal{M}, \boldsymbol{g})$ and a connection $\bar{\boldsymbol{\nabla}}$ on $\mathcal{M}$ — not necessarily the Levi-Civita connection of the metric $\boldsymbol{g}$. Let $\bar{R}^a{}_{bcd}$ denote the Riemann curvature tensor of the connection $\bar{\boldsymbol{\nabla}}$. A ***concomitant*** of $\bar{R}^a{}_{bcd}$ is any tensorial object which can be constructed from the curvature tensor by means of the operations of covariant differentiation and contraction with g_{ab} and g^{ab}. The basic concomitant of $\bar{R}^a{}_{bcd}$ is the ***Ricci tensor***, $\bar{R}_{cd}$, defined by the contraction

$$\bar{R}_{bd} \equiv R^a{}_{bad}.$$

Using the contravariant metric g^{ab} one can define a further concomitant, the ***Ricci scalar*** relative to the metric $\boldsymbol{g}$, $\bar{R}$, as

$$\bar{R} \equiv g^{cd} \bar{R}_{cd}.$$

A concomitant of $\bar{R}^a{}_{bcd}$ which will appear recurrently in this book is the ***Schouten tensor*** relative to $\boldsymbol{g}$, $\bar{L}_{ab}$. In 4 dimensions it is defined as

$$\bar{L}_{ab} = \tfrac{1}{2}\bar{R}_{ab} - \tfrac{1}{12}\bar{R} g_{ab}.$$

The definition of the Schouten tensor is dimension dependent. The expression given above is the appropriate one in 4 dimensions. The definition for 3 dimensions will be discussed in Chapter 13. In the discussion of the spinorial counterpart of the curvature tensor of Chapter (3) a further concomitant arises in a natural way: the ***trace-free Ricci tensor***. In 4 dimensions one has that

$$\bar{\Phi}_{ab} \equiv \bar{R}_{\{ab\}} = \bar{R}_{ab} - \tfrac{1}{4}\bar{R}\bar{g}_{ab}.$$

It is important to observe that the tensors $\bar{R}_{ab}$ and $\bar{L}_{ab}$ are not symmetric unless $\bar{\nabla}$ is a Levi-Civita connection.

Finally, one can define the ***Weyl tensor*** of $\bar{\boldsymbol{\nabla}}$ relative to $\boldsymbol{g}$, $\bar{C}^a{}_{bcd}$ as the fully trace-free part of $R^a{}_{bcd}$. More precisely, one has that

$$\bar{C}_{abcd} \equiv \bar{R}_{\{abcd\}}.$$

The case of a Levi-Civita connection

If $\bar{\boldsymbol{\nabla}}$ is the Levi-Civita connection of the metric $\boldsymbol{g}$, so that $\bar{\boldsymbol{\nabla}} = \boldsymbol{\nabla}$, it can be shown that

$$R^c{}_{dab} = C^c{}_{dab} + 2(\delta^c{}_{[a}L_{d]d} - g_{d[a}L_{b]}{}^c), \qquad (2.19a)$$

$$= C^c{}_{dab} + 2S_{d[a}{}^{ce}L_{b]e}, \qquad (2.19b)$$

where

$$S_{ab}{}^{cd} = \delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c - g_{ab}g^{cd}.$$

This tensor will play a special role in the context of conformal Geometry — see Chapter 5. A spinorial derivation of this decomposition will be provided in Chapter 3.

Some intuition on the origin of decomposition (2.19b) can be provided by considering the so-called ***Kulkarni-Nomizu product***, $\boldsymbol{M} \owedge \boldsymbol{N}$ of two symmetric covariant tensors of rank 2, $\boldsymbol{M}, \boldsymbol{N} \in \mathfrak{T}_2(\mathcal{M})$ which gives rise to a rank 4 covariant tensor with same symmetries of the Riemann tensor of a Levi-Civita connection. In abstract index notation one has that

$$(\boldsymbol{M} \owedge \boldsymbol{N})_{abcd} \equiv M_{ac}N_{bd} + M_{bd}N_{ac} - M_{ad}N_{bc} - M_{bc}N_{ad}. \qquad (2.20)$$

In particular, it can be verified that $\boldsymbol{M} \owedge \boldsymbol{N} = \boldsymbol{N} \owedge \boldsymbol{M}$. A calculation then shows that the decomposition (2.19b) can be rewritten as

$$\boldsymbol{R} = \boldsymbol{L} \owedge \boldsymbol{g} + \boldsymbol{C}, \qquad (2.21)$$

where $\boldsymbol{R}$, $\boldsymbol{L}$ and $\boldsymbol{C}$ denote, respectively, the Riemann, Schouten and Weyl tensors of the Levi-Civita connection $\boldsymbol{\nabla}$. Equation (2.21) can be read as an analogue of the division algorithm for the Kulkarni-Nomizu product —see e.g. Besse (2008).

The Einstein tensor

An important concomitant of the Riemann tensor of a Levi-Civita connection $\boldsymbol{\nabla}$ is the ***Einstein tensor G*** defined by

$$G_{ab} \equiv R_{ab} - \tfrac{1}{2}Rg_{ab}.$$

A well known calculation using the contracted form of the second Bianchi identity, equation (2.11) yields

$$\nabla^a G_{ab} = 0.$$

That is, the Einstein tensor is divergence-free. For further details see e.g. Choquet-Bruhat et al. (1982).

2.5.3 Volume forms and Hodge duals

The spacetime ***volume form*** associated to the metric $\boldsymbol{g}$, ϵ_{abcd}, is defined by the conditions

$$\epsilon_{abcd} = \epsilon_{[abcd]}, \qquad \epsilon_{abcd}\epsilon^{abcd} = -24,$$

and

$$\epsilon_{abcd}e_{\boldsymbol{0}}{}^a e_{\boldsymbol{1}}{}^b e_{\boldsymbol{2}}{}^c e_{\boldsymbol{3}}{}^d = 1,$$

where $\{\boldsymbol{e_a}\}$ is a $\boldsymbol{g}$-orthonormal frame. A spacetime $(\mathcal{M}, \boldsymbol{g})$ has a non-vanishing volume element if and only if $\mathcal{M}$ is orientable —see e.g. O'Neill (1983); Willmore (1993). The following properties can be verified directly:

$$\epsilon_{abcd}\epsilon^{pqrs} = -24\delta_a{}^{[p}\delta_b{}^q\delta_c{}^r\delta_d{}^{s]}, \tag{2.22a}$$
$$\epsilon_{abcd}\epsilon^{pqrd} = -6\delta_a{}^{[p}\delta_b{}^q\delta_c{}^{r]}, \tag{2.22b}$$
$$\epsilon_{abcd}\epsilon^{pqcd} = -4\delta_a{}^{[p}\delta_b{}^{q]}, \tag{2.22c}$$
$$\epsilon_{abcd}\epsilon^{pbcd} = -6\delta_a{}^p. \tag{2.22d}$$

If ∇ denotes the Levi-Civita covariant derivative of the metric $\boldsymbol{g}$ used to define the volume form ϵ_{abcd}, one can then readily verify that $\nabla_a\epsilon_{bcde} = 0$. That is, the volume form is compatible with the Levi-Civita connection of the metric $\boldsymbol{g}$.

The Hodge duals

Given an antisymmetric tensor $F_{ab} = F_{[ab]}$, one can use the volume for to define its ***Hodge dual***, ${}^*F_{ab}$ by

$${}^*F_{ab} \equiv \tfrac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}.$$

This definition can be naturally extended to any tensor with a pair of anti-symmetric indices. Using the identity (2.22c) one readily finds that

$$^{**}F_{ab} = -F_{ab}.$$

Of special relevance are the Hodge duals of the Riemann and Weyl tensors. If R_{abcd} denotes the Riemann curvature of the Levi-Civita covariant derivative ∇, then one can define a ***left dual*** and a ***right dual***, respectively, by

$$^{*}R_{abcd} \equiv \tfrac{1}{2}\epsilon_{ab}{}^{pq}R_{pqcd}, \qquad R^{*}_{abcd} \equiv \tfrac{1}{2}\epsilon_{cd}{}^{pq}R_{abpq}.$$

The Hodge dual can be used to recast the Bianchi identities in an alternative way. More precisely, one has that

$$R_{a[bcd]} = \delta_{[b}{}^{p}\delta_{c}{}^{q}\delta_{d]}{}^{r}R_{apqr} = -\tfrac{1}{6}\epsilon_{sbcd}\left(\epsilon^{spqr}R_{apqr}\right) = -\tfrac{1}{3}\epsilon_{sbcd}R^{*}{}_{ap}{}^{sp}.$$

Thus, the first Bianchi identity $R_{a[bcd]} = 0$ is equivalent to

$$R^{*}_{ab}{}^{cb} = 0. \tag{2.23}$$

Furthermore,

$$\tfrac{1}{2}\epsilon_{f}{}^{abc}\nabla_{[a}R^{d}{}_{|e|bc]} = \nabla_{a}\left(\tfrac{1}{2}\epsilon_{f}{}^{abc}R^{d}{}_{ebc}\right) = \nabla_{a}R^{*d}{}_{ef}{}^{a}.$$

Thus, one has that

$$\nabla^{a}R^{*}{}_{abcd} = 0.$$

Finally, it is noticed that the duals of the Weyl tensor satisfy

$$^{*}C_{abcd} = C^{*}_{abcd}.$$

Further details on the calculations required to obtain all of the above properties can be found in Penrose and Rindler (1984).

2.6 Frame formalisms

Frame formalisms have been used in many areas of Relativity to analyse the properties of the Einstein field equations and their solutions —see e.g. Ellis and van Elst (1998); Ellis et al. (2012); Wald (1984). One of the advantages of the use of a frame formalism is that it leads to consider scalar objects and equations —which are, in general, simpler to manipulate than their tensorial counterparts. A further advantage of the use of frames is that it leads to a straight-forward transcription of tensorial expressions into spinors —see Chapter 3.

The purpose of this chapter is to develop and fix the conventions of a frame to be used in the rest of this book. In doing this, the conventions used in Friedrich (2004) are followed.

2.6.1 Basic definitions and conventions

Given a spacetime $(\mathcal{M}, \boldsymbol{g})$, in what follows let $\{\boldsymbol{e_a}\}$ denote a frame, and let $\{\omega^{\boldsymbol{b}}\}$ denote its dual coframe basis. *For the time being, this frame is not assumed to be $\boldsymbol{g}$-orthogonal.* By definition one has that

$$\langle \boldsymbol{\omega}^{\boldsymbol{b}}, \boldsymbol{e_a} \rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}. \tag{2.24}$$

In what follows, it will be assumed one has a connection $\boldsymbol{\nabla}$ *which, for the time being, is assumed to be general* —that is, it is not necessarily metric or torsion-free. The ***connection coefficients*** of $\boldsymbol{\nabla}$ with respect to the frame $\boldsymbol{e_a}$, $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$, are defined via

$$\nabla_{\boldsymbol{a}} \boldsymbol{e_b} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \boldsymbol{e_c}, \tag{2.25}$$

where $\nabla_{\boldsymbol{a}} \equiv e_{\boldsymbol{a}}{}^a \nabla_a$ denotes the ***covariant directional derivative*** in the direction of $e_{\boldsymbol{a}}$. As $\nabla_{\boldsymbol{a}} \boldsymbol{e_b}$ is a vector, it follows then that

$$\langle \omega^{\boldsymbol{c}}, \nabla_{\boldsymbol{a}} \boldsymbol{e_b} \rangle = \langle \omega^{\boldsymbol{c}}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} e_{\boldsymbol{d}} \rangle = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} \langle \omega^{\boldsymbol{c}}, e_{\boldsymbol{d}} \rangle = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}.$$

The latter expression could have been used, alternatively, as a definition of the connection coefficients. In order to carry out computations one also needs an expression for $\nabla_{\boldsymbol{a}} \omega^{\boldsymbol{b}}$. By analogy with (2.25) write $\nabla_{\boldsymbol{a}} \omega^{\boldsymbol{b}} = \mho_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \omega^{\boldsymbol{c}}$. The coefficients $\mho_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \omega^{\boldsymbol{c}}$ can be expressed in terms of the connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ by differentiating (2.24) with respect to $\nabla_{\boldsymbol{d}}$. Noting that $\delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$ is constant scalar, one has on one hand that

$$\nabla_{\boldsymbol{d}}(\langle \boldsymbol{\omega}^{\boldsymbol{b}}, \boldsymbol{e_a} \rangle) = \boldsymbol{e_d}(\langle \boldsymbol{\omega}^{\boldsymbol{b}}, \boldsymbol{e_a} \rangle) = \boldsymbol{e}_d(\delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}) = 0,$$

while on the other

$$\nabla_{\boldsymbol{d}}(\langle \boldsymbol{\omega}^{\boldsymbol{b}}, \boldsymbol{e_a} \rangle) = \langle \nabla_{\boldsymbol{d}} \omega^{\boldsymbol{b}}, \boldsymbol{e_a} \rangle + \langle \omega^{\boldsymbol{b}}, \nabla_{\boldsymbol{d}} \boldsymbol{e_a} \rangle = \left(\mho_{\boldsymbol{d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} + \Gamma_{\boldsymbol{d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \right) \langle \omega^{\boldsymbol{c}}, \boldsymbol{e_a} \rangle,$$

so that $\mho_{\boldsymbol{d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = -\Gamma_{\boldsymbol{d}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$. Consequently, one has

$$\nabla_{\boldsymbol{a}} \omega^{\boldsymbol{b}} = -\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} \omega^{\boldsymbol{c}}. \tag{2.26}$$

It is observed that the specification of the 4^3 connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ fully determines the connection $\boldsymbol{\nabla}$ —a generalisation of this argument shows that every manifold admits a connection —see e.g. Willmore (1993).

Consider now $\boldsymbol{v} \in T(\mathcal{M})$ and $\boldsymbol{\alpha} \in T^*(\mathcal{M})$. Writing the above, respectively, in terms of the frame and coframe one has

$$\boldsymbol{v} = v^{\boldsymbol{a}} \boldsymbol{e}_{\boldsymbol{a}}, \qquad v^{\boldsymbol{a}} \equiv \langle \boldsymbol{\omega}^{\boldsymbol{a}}, \boldsymbol{v} \rangle,$$
$$\boldsymbol{\alpha} = \alpha_{\boldsymbol{a}} \omega^{\boldsymbol{a}}, \qquad \alpha_{\boldsymbol{a}} \equiv \langle \boldsymbol{\alpha}, \boldsymbol{e}_{\boldsymbol{a}} \rangle.$$

In order to further develop the frame formalism it will be convenient to define

$$\nabla_{\boldsymbol{a}} v^{\boldsymbol{b}} \equiv \langle \boldsymbol{\omega}^{\boldsymbol{b}}, \nabla_{\boldsymbol{a}} \boldsymbol{v} \rangle, \qquad \nabla_{\boldsymbol{a}} \alpha_{\boldsymbol{b}} \equiv \langle \nabla_{\boldsymbol{a}} \boldsymbol{\alpha}, \boldsymbol{e}_{\boldsymbol{b}} \rangle.$$

It follows then from (2.25) and (2.26) that

$$\nabla_{\boldsymbol{a}} v^{\boldsymbol{b}} = \boldsymbol{e}_{\boldsymbol{a}}(v^{\boldsymbol{b}}) + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} v^{\boldsymbol{c}}, \qquad \nabla_{\boldsymbol{a}} \alpha_{\boldsymbol{b}} = \boldsymbol{e}(\alpha_{\boldsymbol{b}}) - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \alpha_{\boldsymbol{c}}. \tag{2.27}$$

The above expressions extend in the obvious way to higher rank components. Notice, in particular, that

$$\nabla_{\boldsymbol{a}} \delta_{\boldsymbol{b}}{}^{\boldsymbol{c}} = -\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} \delta_{\boldsymbol{d}}{}^{\boldsymbol{c}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} \delta_{\boldsymbol{b}}{}^{\boldsymbol{d}} = -\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0.$$

Metric connections

Now assume that the connection $\boldsymbol{\nabla}$ is $\boldsymbol{g}$-compatible (i.e. $\boldsymbol{\nabla g} = 0$) and that the frame $\{\boldsymbol{e}_a\}$ is $\boldsymbol{g}$-orthogonal —that is, $\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}) = \eta_{\boldsymbol{ab}}$. It follows then that

$$\nabla_{\boldsymbol{a}} \left(\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{b}}, \boldsymbol{e}_{\boldsymbol{c}}) \right) = \boldsymbol{e}_{\boldsymbol{a}}(\eta_{\boldsymbol{bc}}) = 0,$$

and that

$$\nabla_{\boldsymbol{a}} \boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{b}}, \boldsymbol{e}_{\boldsymbol{c}}) = \boldsymbol{g}(\nabla_{\boldsymbol{a}} \boldsymbol{e}_{\boldsymbol{b}}, \boldsymbol{e}_{\boldsymbol{c}}) + \boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{b}}, \nabla_{\boldsymbol{a}} \boldsymbol{e}_{\boldsymbol{c}}).$$

Thus, using (2.25) one concludes that

$$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} \eta_{\boldsymbol{dc}} + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} \eta_{\boldsymbol{bd}} = 0. \tag{2.28}$$

Finally, it is recalled that in the case of a Levi-Civita connection and with the choice of a coordinate basis $\{\partial/\partial x^\mu\}$, the Koszul formula, equation (2.18), shows that the connection coefficients reduce to the classical expression for the Christoffel symbols:

$$\Gamma_\mu{}^\nu{}_\lambda = \tfrac{1}{2} g^{\nu\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}).$$

2.6.2 Frame description of the torsion and curvature

Following the spirit of the previous subsections, let

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \equiv \boldsymbol{e_a}{}^a \boldsymbol{e_b}{}^b \boldsymbol{\omega}^{\boldsymbol{c}}{}_c \Sigma_a{}^c{}_b,$$

denote the ***components of the torsion tensor*** $\Sigma_a{}^c{}_b$ with respect to $\boldsymbol{e_a}$ and $\boldsymbol{\omega}^a$. Given $f \in \mathfrak{X}(\mathcal{M})$, a computation then shows that

$$\begin{aligned}\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c}(f) &= \nabla_{\boldsymbol{a}}\boldsymbol{e_b}(f) - \nabla_{\boldsymbol{b}}\boldsymbol{e_a}(f) \\ &= (\boldsymbol{e_a}\boldsymbol{e_b}(f) - \Gamma_{\boldsymbol{a}}{}^{c}{}_{\boldsymbol{b}}\boldsymbol{e}_c(f)) + (\boldsymbol{e_b}\boldsymbol{e_a}(f) - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}}\boldsymbol{e}_c(f)) \\ &= [\boldsymbol{e_a}, \boldsymbol{e_b}](f) - (\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})\,\boldsymbol{e_c}(f),\end{aligned}$$

where it has been used that $\nabla_{\boldsymbol{a}}f = \boldsymbol{e_a}(f)$.

In order to obtain a frame description of the Riemann curvature tensor one makes use of equation (2.8) with $v^c = e_{\boldsymbol{d}}{}^c$, and contracts with $e_{\boldsymbol{a}}{}^a e_{\boldsymbol{b}}{}^b \omega^{\boldsymbol{c}}{}_d$. One then has that

$$R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv e_{\boldsymbol{a}}{}^a e_{\boldsymbol{b}}{}^b \omega^{\boldsymbol{c}}{}_d R^c{}_{dab}.$$

Furthermore, one can compute

$$\begin{aligned}e_{\boldsymbol{a}}{}^a e_{\boldsymbol{b}}{}^b \omega^{\boldsymbol{c}}{}_c \nabla_a \nabla_b e_{\boldsymbol{d}}{}^c &= \omega^{\boldsymbol{c}}{}_c \nabla_{\boldsymbol{a}}(\nabla_{\boldsymbol{b}} e_{\boldsymbol{d}}{}^c) - \omega^{\boldsymbol{c}}{}_c(\nabla_{\boldsymbol{a}} e_{\boldsymbol{b}}{}^b)(\nabla_b e_{\boldsymbol{d}}{}^c), \\ &= \omega^{\boldsymbol{c}}{}_c \nabla_{\boldsymbol{a}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} e_{\boldsymbol{f}}{}^c) - \omega^{\boldsymbol{c}}{}_c \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}} \nabla_{\boldsymbol{f}} e_{\boldsymbol{d}}{}^c \\ &= \omega^{\boldsymbol{c}}{}_c \boldsymbol{e_a}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}) e_{\boldsymbol{f}}{}^c + \omega^{\boldsymbol{c}}{}_c \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \nabla_{\boldsymbol{a}} e_{\boldsymbol{f}}{}^c - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}} \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} \\ &= \boldsymbol{e_a}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}} \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}.\end{aligned}$$

A similar computation can be carried out for $e_{\boldsymbol{a}}{}^a e_{\boldsymbol{b}}{}^b \omega^{\boldsymbol{c}}{}_c \nabla_b \nabla_a e_{\boldsymbol{d}}{}^c$ so that one obtains

$$\begin{aligned}R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = \boldsymbol{e_a}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \boldsymbol{e_b}(\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}) \\ + \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}} \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}} \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}. \qquad (2.29)\end{aligned}$$

2.7 Congruences and submanifolds

The formulation of an initial value problem in General Relativity requires the decomposition of tensorial objects in terms of *temporal* and *spatial components*. This decomposition requires, in turn, an understanding of the way geometric structures of the spacetime are inherited by suitable subsets thereof. For concreteness, in what follows a spacetime $(\mathcal{M}, \boldsymbol{g})$ is assumed. Hence $\mathcal{M}$ is a 4-dimensional manifold and $\boldsymbol{g}$ denotes a Lorentzian metric.

2.7.1 Basic notions

Submanifolds

Intuitively, ***submanifold*** of $\mathcal{M}$ is a set $\mathcal{N} \subset \mathcal{M}$ which inherits a manifold structure from $\mathcal{M}$. A more precise definition of submanifolds requires the concept of embedding. Given two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$ an ***embedding*** is a map $\varphi : \mathcal{N} \to \mathcal{M}$ such that:

(a) the push-forward $\varphi^* : T_p(\mathcal{N}) \to T_{\varphi(p)}(\mathcal{M})$ is injective for every point $p \in \mathcal{M}$;
(b) the manifold $\mathcal{N}$ is diffeomorphic to the image $\varphi(\mathcal{N})$.

In terms of the above, one defines a ***submanifold*** $\mathcal{N}$ of $\mathcal{M}$ as the image, $\varphi(\mathcal{S}) \subset \mathcal{M}$, of a k-dimensional manifold $\mathcal{S}$ $(k < 4)$ by an embedding $\varphi : \mathcal{S} \to \mathcal{M}$. In several contexts it is convenient to identify $\mathcal{N}$ with $\varphi(\mathcal{N})$ and denote, in an abuse of notation, both manifolds by $\mathcal{N}$. A 3-dimensional submanifold of $\mathcal{M}$ is called an ***hypersurface***. In what follows, a generic hypersurface will be denoted by $\mathcal{S}$. As a consequence of its manifold structure, one can associate to $\mathcal{S}$ a tangent and cotangent bundles, $T(\mathcal{S})$ and $T^*(\mathcal{S})$, and more generally, a tensor bundle $\mathfrak{T}^\bullet(\mathcal{S})$.

A vector $\boldsymbol{u}$ (u^α) on $\mathcal{S}$ can be associated to a vector of $\mathcal{M}$ by the push-forward $\varphi_*\boldsymbol{u}$. A vector on $\boldsymbol{v}$ on $\mathcal{M}$ is said to be ***normal*** to $\mathcal{S}$ if $\boldsymbol{g}(\boldsymbol{v}, \varphi_*\boldsymbol{u}) = 0$. If $\epsilon \equiv \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = \pm 1$, one speaks of a ***unit normal vector*** —in this case the surface is said to be timelike if $\epsilon = -1$ and spacelike is $\epsilon = 1$. A hypersurface $\mathcal{S}$ of a Lorentzian manifold $\mathcal{M}$ is orientable if and only if there exists a unique smooth normal vector field on $\mathcal{S}$ —see e.g. O'Neill (1983).

A natural way of specifying a hypersurface is as the level surface of some function $f \in \mathfrak{X}(\mathcal{M})$. In this case one has that the gradient $\mathbf{d}f \in T^*(\mathcal{M})$ gives rise to a normal vector $(\mathbf{d}f)^\sharp \in T(\mathcal{M})$. The ***normal*** of $\mathcal{S}$, (ν_a) is then defined as a unit 1-form in the direction of $\mathbf{d}f$ —i.e. $\boldsymbol{g}^\sharp(\boldsymbol{\nu}, \boldsymbol{\nu}) = \epsilon$. The normal is defined in the restriction to $\mathcal{S}$ of the cotangent bundle $T^*(\mathcal{M})$. In the case of a spacelike hypersurface, the normal constructed in this way is taken, conventionally, to be future pointing.

Foliations

By a ***foliation*** of a spacetime $(\mathcal{M}, \boldsymbol{g})$ its is understood a family, $\{\mathcal{S}_t\}_{t\in\mathbb{R}}$, of *spacelike* hypersurfaces $\mathcal{S}_t$, such that

$$\bigcup_{t\in\mathbb{R}} \mathcal{S}_t = \mathcal{M}, \qquad \mathcal{S}_{t_1} \cap \mathcal{S}_{t_2} = \emptyset \quad \text{for} \quad t_1 \neq t_2.$$

The hypersurfaces $\mathcal{S}_t$ are called the ***leaves*** or ***slices*** of the foliation. The foliation $\{\mathcal{S}_t\}_{t\in\mathbb{R}}$ can be defined in terms of a scalar field $f \in \mathfrak{X}(\mathcal{M})$ such that the leaves of the foliation are level surfaces of f. That is, given $p \in \mathcal{S}_t$, then $f(p) = t$. The scalar field f is said to be a ***time function***. In what follows, it will be convenient to identify f and t. The ***normal of a foliation*** is a normalised vector field, $\boldsymbol{\nu}$, orthogonal to each leave of a foliation. The gradient $\mathbf{d}t$ provides a further 1-form normal to the leaves. In general one has that

$$\boldsymbol{\nu} = N\mathbf{d}t.$$

The proportionality factor N is called the ***lapse*** of the foliation.

Distributions

A ***distribution*** Π is an assignment at each $p \in \mathcal{M}$ of a k-dimensional subspace Π_p of the tangent space $T_p(\mathcal{M})$. A submanifold $\mathcal{N}$ of $\mathcal{M}$ such that $\Pi_p = T_p(\mathcal{S})$ for all $p \in \mathcal{S}$ is said to be an ***integrable manifold*** of Π. If for every $p \in \mathcal{M}$ there is an integrable manifold, then Π is said to be ***integrable***. One has the following classical result —see e.g. Choquet-Bruhat et al. (1982) for details:

Theorem 1 (Frobenius theorem) *A distribution Π on $\mathcal{M}$ is integrable if and only if for $\boldsymbol{u}$, $\boldsymbol{v} \in \Pi$, one has $[\boldsymbol{u}, \boldsymbol{v}] \in \Pi$.*

2.7.2 Geometry of congruences

Integral curves

A curve $\gamma : I \to \mathcal{M}$ is the ***integral curve*** of a vector $\boldsymbol{v}$ if the tangent vector of the curve γ coincides with $\boldsymbol{v}$. The standard theorems of the theory of ordinary differential equations —see e.g. Hartman (1987)— ensure that given $\boldsymbol{v} \in T(\mathcal{M})$, for all $p \in \mathcal{M}$ there exists an interval $I \ni 0$ and a unique integral curve $\gamma : I \to \mathcal{M}$ of $\boldsymbol{v}$ such that $\gamma(0) = p$. Moreover, if two integral curves $\gamma_1, \gamma_2 : I \to \mathcal{M}$ are such that $\gamma_1(t) = \gamma_2(t)$ for some $t \in I$, then $\gamma_1 = \gamma_2$. The collection of all the integral curves of $\boldsymbol{v}$ that start at $p \in \mathcal{M}$ define a single integral curve, the ***maximal integral curve***. If the domain of an integral curve is $\mathbb{R}$, then the integral curve is said to be ***complete***.

Congruences

The notion of a *congruence of geodesics* has been discussed in Section 2.4.5. More generally, a ***congruence of curves*** is the set of integral curves of a (nowhere vanishing) vector field $\boldsymbol{v}$ on $\mathcal{M}$. In the remaining of this section *it will be assumed that the curves of a congruence are non-intersecting and timelike.* This will be the case of most relevance in this book. In what follows, $\boldsymbol{t}$ will denote the vector field generating a timelike congruence. Without loss of generality it is assumed that $\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{t}) = 1$.

As in previous sections let $\{\boldsymbol{e_a}\}$ denote a $\boldsymbol{g}$-orthonormal frame. The orthonormal frame can be adapted to the congruence defined by the vector field

$\boldsymbol{t}$ by setting $\boldsymbol{e_0} = \boldsymbol{t}$. Given a point $p \in \mathcal{M}$, the tangent space $T_p(\mathcal{M})$ is naturally split in a part tangential to $\boldsymbol{t}$, to be denote by $\langle \boldsymbol{t} \rangle|_p$ (the 1-dimensional subspace spanned by $\boldsymbol{t}$), and a part which orthogonal to it which will be denoted by $\langle \boldsymbol{t} \rangle^\perp|_p = \langle \boldsymbol{e}_\alpha \rangle|_p$ (the 3-dimensional subspace generated by $\{\boldsymbol{e_\alpha}\}$ with $\boldsymbol{\alpha} = \mathbf{1}, \mathbf{2}, \mathbf{3}$). One writes then

$$T_p(\mathcal{M}) = \langle \boldsymbol{t} \rangle|_p \oplus \langle \boldsymbol{t} \rangle^\perp|_p, \tag{2.30}$$

where $\oplus$ denotes the ***direct sum*** of vectorial spaces —that is, any vector in $T_p\mathcal{M}$ can be written in a unique way as the sum of an element in $\langle \boldsymbol{t} \rangle|_p$ and an element in $\langle \boldsymbol{t} \rangle^\perp|_p$. Hence, one sees that *the congruence generated by* $\boldsymbol{t}$ *gives rise to a 3-dimensional distribution* Π. At every point $p \in \mathcal{M}$, the subspace $\Pi_p \subset T_p(\mathcal{M})$ corresponds to $\langle \boldsymbol{e_\alpha} \rangle|_p$ —that is, $\{\boldsymbol{e_\alpha}\}$ is a basis of Π_p. In the sequel, $\langle \boldsymbol{t} \rangle$ and $\langle \boldsymbol{t} \rangle^\perp$ will denote, respectively, the *disjoint union* of all the spaces $\langle \boldsymbol{t} \rangle|_p$ and $\langle \boldsymbol{t} \rangle|_p^\perp$, $p \in \mathcal{M}$. According to this notation one has that $\Pi = \langle \boldsymbol{t} \rangle^\perp$. Frobenius Theorem 1 gives the necessary and sufficient conditions for the distribution defined by $\langle \boldsymbol{t} \rangle|_p^\perp$ to be integrable —that is, for the vector $\boldsymbol{t}$ to be the unit normal of a foliation $\{\mathcal{S}_t\}$ of spacetime.

Making use of $\boldsymbol{g}^\sharp$ one obtains an analogous decomposition for the cotangent space. Namely, one has that

$$T_p^*(\mathcal{M}) = \langle \boldsymbol{t}^\flat \rangle|_p \oplus \langle \boldsymbol{t}^\flat \rangle^\perp|_p, \tag{2.31}$$

with $\langle \boldsymbol{t}^\flat \rangle^\perp|_p = \langle \boldsymbol{\omega_\alpha} \rangle|_p$. The decompositions (2.30) and (2.31) can be extended in the natural way to higher rank tensors by considering tensor products. Given a tensor T_{ab} with components with respect to the frame $\{\boldsymbol{e_a}\}$ given by $T_{\boldsymbol{ab}}$, one has that $T_{\boldsymbol{\alpha\beta}} \equiv e_{\boldsymbol{\alpha}}{}^a e_{\boldsymbol{\beta}}{}^b T_{ab}$ and $T_{\mathbf{00}} \equiv t^a t^b T_{ab}$ correspond, respectively, to the components of T_{ab} ***transversal*** and ***longitudinal*** to $\boldsymbol{t}$; finally $T_{\mathbf{0}\boldsymbol{\alpha}} \equiv t^a e_{\boldsymbol{\alpha}}{}^a T_{ab}$ and $T_{\boldsymbol{\alpha}\mathbf{0}} \equiv e_{\boldsymbol{\alpha}}{}^a t^b T_{ab}$ are ***mixed transversal-longitudinal*** components.

The covariant derivative of $\boldsymbol{t}$

In order to further discuss the geometry of the congruence generated by the timelike vector $\boldsymbol{t}$ it is convenient to introduce the so-called ***Weingarten map*** $\boldsymbol{\chi} : \langle \boldsymbol{t} \rangle^\perp \to \langle \boldsymbol{t} \rangle^\perp$ defined by

$$\boldsymbol{\chi}(\boldsymbol{u}) \equiv \nabla_{\boldsymbol{u}} \boldsymbol{t}, \qquad \boldsymbol{u} \in \langle \boldsymbol{t} \rangle^\perp.$$

One can readily verify that

$$\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{\chi}(\boldsymbol{u})) = \boldsymbol{g}(\boldsymbol{t}, \nabla_{\boldsymbol{u}} \boldsymbol{t}) = \tfrac{1}{2} \nabla_{\boldsymbol{u}} (\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{t})) = 0, \tag{2.32}$$

so that indeed $\boldsymbol{\chi}(\boldsymbol{u}) \in \langle \boldsymbol{t}\rangle^{\perp}$. Hence, it is enough to consider the Weingarten map evaluated on a basis $\{\boldsymbol{e}_{\boldsymbol{\alpha}}\}$ of $\langle \boldsymbol{t}\rangle^{\perp}$. Accordingly, one defines

$$\boldsymbol{\chi}_{\boldsymbol{\alpha}} \equiv \boldsymbol{\chi}(\boldsymbol{e}_{\boldsymbol{\alpha}}) = \chi_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{e}_{\beta}, \qquad \chi_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \equiv \langle \boldsymbol{\omega}^{\beta}, \boldsymbol{\chi}_{\boldsymbol{\alpha}}\rangle.$$

In the sequel, it will be more convenient to work with $\chi_{\boldsymbol{\alpha\beta}} \equiv \eta_{\boldsymbol{\beta\gamma}}\chi_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}$. The scalars $\chi_{\boldsymbol{\alpha\beta}}$ can be considered as the components of a rank 2 covariant tensor on $\boldsymbol{\chi} \in \langle \boldsymbol{t}\rangle^{\perp} \otimes \langle \boldsymbol{t}\rangle^{\perp}$ —the ***Weingarten tensor*** of the congruence. The symmetric part $\theta_{\boldsymbol{\alpha\beta}} \equiv \chi_{(\boldsymbol{\alpha\beta})}$ and the antisymmetric part $\omega_{\boldsymbol{\alpha\beta}} \equiv \chi_{[\boldsymbol{\alpha\beta}]}$ are called is called the ***expansion*** and the ***twist*** of the congruence. From $\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{e}_{\boldsymbol{\alpha}}) = 0$ it follows that $\boldsymbol{g}(\nabla_{\boldsymbol{\beta}}\boldsymbol{t}, \boldsymbol{e}_{\boldsymbol{\alpha}}) = -\boldsymbol{g}(\boldsymbol{t}, \nabla_{\boldsymbol{\beta}}\boldsymbol{e}_{\boldsymbol{\alpha}})$. Hence, one can compute

$$\begin{aligned}
\chi_{\boldsymbol{\alpha\beta}} &= \boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{\alpha}}, \boldsymbol{\chi}_{\boldsymbol{\beta}}) = \boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{\alpha}}, \nabla_{\boldsymbol{\beta}}\boldsymbol{t}) = -\boldsymbol{g}(\boldsymbol{t}, \nabla_{\boldsymbol{\beta}}\boldsymbol{e}_{\boldsymbol{\alpha}})\\
&= -\boldsymbol{g}(\boldsymbol{t}, \nabla_{\boldsymbol{\alpha}}\boldsymbol{e}_{\beta} - [\boldsymbol{e}_{\boldsymbol{\alpha}}, \boldsymbol{e}_{\boldsymbol{\beta}}]) = \boldsymbol{g}(\nabla_{\boldsymbol{\alpha}}\boldsymbol{t}, \boldsymbol{e}_{\beta}) + \boldsymbol{g}(\boldsymbol{t}, [\boldsymbol{e}_{\boldsymbol{\alpha}}, \boldsymbol{e}_{\boldsymbol{\beta}}])\\
&= \boldsymbol{g}(\chi_{\boldsymbol{\alpha}}, \boldsymbol{e}_{\boldsymbol{\beta}}) + \boldsymbol{g}(\boldsymbol{t}, [\boldsymbol{e}_{\boldsymbol{\alpha}}, \boldsymbol{e}_{\boldsymbol{\beta}}])\\
&= \chi_{\boldsymbol{\beta\alpha}} + \boldsymbol{g}(\boldsymbol{t}, [\boldsymbol{e}_{\boldsymbol{\alpha}}, \boldsymbol{e}_{\boldsymbol{\beta}}]).
\end{aligned}$$

where in the third line it has been used that $\nabla_{\boldsymbol{\alpha}}\boldsymbol{e}_{\boldsymbol{\alpha}} - \nabla_{\boldsymbol{\beta}}\boldsymbol{e}_{\boldsymbol{\alpha}} = [\boldsymbol{e}_{\boldsymbol{\alpha}}, \boldsymbol{e}_{\boldsymbol{\beta}}]$ as $\boldsymbol{\nabla}$ is torsion-free. Hence, by Frobenius theorem 1, the symmetry relation $\chi_{\boldsymbol{\alpha\beta}} = \chi_{\boldsymbol{\beta\alpha}}$ if and only if the distribution $\langle \boldsymbol{t}\rangle^{\perp}$ is integrable. The components $\chi_{\boldsymbol{\alpha\beta}}$ are related to the connection coefficients of $\boldsymbol{\nabla}$ as it can be seen from

$$\chi_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \langle \omega^{\boldsymbol{\beta}}, \chi_{\boldsymbol{\alpha}}\rangle = \langle \boldsymbol{\omega}^{\boldsymbol{\beta}}, \nabla_{\boldsymbol{\alpha}}\boldsymbol{e}_0\rangle = \langle \boldsymbol{\omega}^{\boldsymbol{\beta}}, \Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{b}}{}_{\boldsymbol{0}}\boldsymbol{e}_{\boldsymbol{b}}\rangle = \Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}{}_{\boldsymbol{0}}.$$

Alternatively, one has that

$$\chi_{\boldsymbol{\alpha\beta}} = \Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}}\eta_{\boldsymbol{c\beta}} = -\Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{c}}{}_{\boldsymbol{\beta}}\eta_{\boldsymbol{c0}} = -\Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{0}}{}_{\boldsymbol{\beta}},$$

where the last two equalities follow from the metricity of the connection —see equation (2.28).

From the discussion in the previous paragraphs it follows that the components $\nabla_{\boldsymbol{\alpha}}t_{\boldsymbol{0}} \equiv \langle (\nabla_{\boldsymbol{\alpha}}\boldsymbol{t})^{\flat}, \boldsymbol{e}_{\boldsymbol{0}}\rangle$ of the covariant derivative of $\boldsymbol{t}$ vanish. Moreover, using (2.32) it follows that also $\nabla_{\boldsymbol{0}}t_{\boldsymbol{0}} \equiv \langle (\nabla_{\boldsymbol{0}}\boldsymbol{t})^{\flat}, \boldsymbol{e}_0\rangle = 0$. Consequently, one has that the ***acceleration*** of the congruence, $\boldsymbol{a} \equiv \nabla_{\boldsymbol{0}}\boldsymbol{t} = \nabla_{\boldsymbol{0}}\boldsymbol{e}_{\boldsymbol{0}}$, if non-vanishing is spatial —i.e. $\boldsymbol{g}(\boldsymbol{a}, \boldsymbol{t}) = 0$ so that $\boldsymbol{a} \in \langle \boldsymbol{t}\rangle^{\perp}$. Using the definition of connection coefficients of the connection $\boldsymbol{\nabla}$ it follows that

$$\boldsymbol{a}^{\boldsymbol{\alpha}} \equiv \langle \boldsymbol{\omega}^{\boldsymbol{\alpha}}, \boldsymbol{a}\rangle = \Gamma_{\boldsymbol{0}}{}^{\boldsymbol{\alpha}}{}_{\boldsymbol{0}}.$$

2.7.3 Geometry of hypersurfaces

Given a spacetime $(\mathcal{M}, \boldsymbol{g})$ and a hypersurface thereof, $\mathcal{S}$, the embedding $\varphi : \mathcal{S} \to \mathcal{M}$ induces on $\mathcal{S}$ a rank 2 covariant tensor $\boldsymbol{h}$, the ***intrinsic metric*** or ***first fundamental form*** of $\mathcal{S}$ via the pull back of $\boldsymbol{g}$ to $\mathcal{S}$:

$$\boldsymbol{h} \equiv \varphi^* \boldsymbol{g}.$$

As a consequence of the definition of embedding, the intrinsic metric $\boldsymbol{h}$ will be non-degenerate if the hypersurface $\mathcal{S}$ is timelike or spacelike. Its signature will be $(+,-,-)$ in the former case and $(-,-,-)$ in the latter. The (unique) Levi-Civita connection of $\boldsymbol{h}$ will be denoted by $\boldsymbol{D}$. Alternatively, one can define the ***pull-back connection***

$$\varphi^* \boldsymbol{\nabla} : T(\mathcal{S}) \times T(\mathcal{S}) \to T(\mathcal{S})$$

via

$$(\varphi^* \nabla)_{\boldsymbol{v}} \boldsymbol{u} \equiv \varphi_*^{-1} \left(\boldsymbol{\nabla}_{\varphi_* \boldsymbol{v}} (\varphi_* \boldsymbol{u}) \right).$$

It can be readily verified that $\varphi^* \boldsymbol{\nabla}$ as defined above is indeed a linear connection. Given a function $f \in \mathfrak{X}(\mathcal{M})$, the action of $\varphi^* \boldsymbol{\nabla}$ on the pull-back $\varphi^* f$ is defined by

$$(\varphi^* \nabla)_{\boldsymbol{v}} (\varphi^* f) \equiv \varphi^* (\nabla_{\varphi^* \boldsymbol{v}} f).$$

Discuss the inverse of the push-forward

In order to define the action of $\varphi^* \boldsymbol{\nabla}$ on 1-forms, one requires the Leibnitz rule

$$(\varphi^* \nabla)_{\boldsymbol{v}} \langle \varphi^* \boldsymbol{\omega}, \boldsymbol{u} \rangle = \langle (\varphi^* \nabla)_{\boldsymbol{v}} (\varphi^* \boldsymbol{\omega}), \boldsymbol{u} \rangle + \langle \varphi^* \boldsymbol{\omega}, (\varphi^* \nabla)_{\boldsymbol{v}} \boldsymbol{u} \rangle,$$

for $\boldsymbol{\omega} \in T^*(\mathcal{M})$ and $\boldsymbol{u}$, $\boldsymbol{v} \in T(\mathcal{S})$. As the connection $\boldsymbol{\nabla}$ also satisfies the Leibnitz rule one also has that

$$\begin{aligned}
\boldsymbol{\nabla}_{\varphi_* \boldsymbol{v}} \langle \omega, \varphi_* \boldsymbol{u} \rangle &= \langle \boldsymbol{\nabla}_{\varphi_* \boldsymbol{v}} \boldsymbol{\omega}, \varphi_* \boldsymbol{u} \rangle + \langle \boldsymbol{\omega}, \boldsymbol{\nabla}_{\varphi_* \boldsymbol{v}} \varphi_* \boldsymbol{u} \rangle \\
&= \langle \boldsymbol{\nabla}_{\varphi_* \boldsymbol{v}} \boldsymbol{\omega}, \varphi_* \boldsymbol{u} \rangle + \langle \boldsymbol{\omega}, \varphi_* \circ \varphi_*^{-1} (\boldsymbol{\nabla}_{\varphi_* \boldsymbol{v}} \varphi_* \boldsymbol{u}) \rangle \\
&= \langle \boldsymbol{\nabla}_{\varphi_* \boldsymbol{v}} \boldsymbol{\omega}, \varphi_* \boldsymbol{u} \rangle + \langle \varphi^* \boldsymbol{\omega}, \varphi_*^{-1} \boldsymbol{\nabla}_{\varphi_* \boldsymbol{v}} \varphi_* \boldsymbol{u} \rangle \\
&= \langle \boldsymbol{\nabla}_{\varphi_* \boldsymbol{v}} \boldsymbol{\omega}, \varphi_* \boldsymbol{u} \rangle + \langle \varphi^* \boldsymbol{\omega}, (\varphi^* \boldsymbol{\nabla})_{\boldsymbol{v}} \boldsymbol{u} \rangle.
\end{aligned}$$

Now, one also has that

$$\varphi^* (\boldsymbol{\nabla}_{\varphi_* \boldsymbol{v}} \langle \boldsymbol{\omega}, \varphi_* \boldsymbol{u} \rangle) = (\varphi^* \boldsymbol{\nabla})_{\boldsymbol{v}} \langle \boldsymbol{\omega}, \varphi_* \boldsymbol{u} \rangle = (\varphi^* \boldsymbol{\nabla})_{\boldsymbol{v}} \langle \varphi^* \boldsymbol{\omega}, \boldsymbol{u} \rangle.$$

A comparison of the above expressions suggests defining

$$(\varphi^* \nabla)_{\boldsymbol{v}} \varphi^* \boldsymbol{\omega} \equiv \varphi^* (\nabla_{\varphi_* \boldsymbol{v}} \boldsymbol{\omega}).$$

In a natural way, the embedding $\varphi : \mathcal{S} \to \mathcal{M}$ takes the connection $\boldsymbol{\nabla}$ to the connection $\boldsymbol{D}$. More precisely, one has the following result:

Lemma 2 *Given* $\boldsymbol{u}$, $\boldsymbol{w} \in T(\mathcal{S})$

$$\varphi_* (D_{\boldsymbol{w}} \boldsymbol{u}) = \nabla_{\varphi_* \boldsymbol{w}} (\varphi_* \boldsymbol{u}). \tag{2.33}$$

Proof This last expression is now used to verify that the pull-back connection $\varphi^*\boldsymbol{\nabla}$ is torsion-free. Given a function $f \in \mathfrak{X}(\mathcal{M})$ one has that

$$\begin{aligned}(\varphi^*\boldsymbol{\nabla})_{\boldsymbol{u}}\left((\varphi^*\boldsymbol{\nabla})_{\boldsymbol{v}}(\varphi^* f)\right) &= (\varphi^*\boldsymbol{\nabla})_{\boldsymbol{u}}\left(\varphi^*(\boldsymbol{\nabla}_{\varphi^*\boldsymbol{v}} f)\right)\\ &= \varphi^*\left(\boldsymbol{\nabla}_{\varphi_*\boldsymbol{u}}\nabla_{\varphi_*\boldsymbol{v}} f\right)\\ &= \varphi^*\left(\nabla_{\varphi_*\boldsymbol{v}}\boldsymbol{\nabla}_{\varphi_*\boldsymbol{u}} f\right)\\ &= (\varphi^*\boldsymbol{\nabla})_{\boldsymbol{v}}\left((\varphi^*\boldsymbol{\nabla})_{\boldsymbol{u}}(\varphi^* f)\right),\end{aligned}$$

where to pass from the second to the third line it has been used that the connection $\boldsymbol{\nabla}$ is torsion-free. One thus concludes that the connection $\varphi^*\boldsymbol{\nabla}$ is indeed torsion free. Finally, it can be readily verified that on has compatibility with the metric $\boldsymbol{h}$. Indeed,

$$(\varphi^*\boldsymbol{\nabla})_{\boldsymbol{v}}\boldsymbol{h} = (\varphi^*\boldsymbol{\nabla})_{\boldsymbol{v}}(\varphi^*\boldsymbol{g}) = \varphi^*(\nabla_{\varphi_*\boldsymbol{v}}\boldsymbol{g}) = 0,$$

where the las equality follows from the $\boldsymbol{g}$-compatibility of the connection $\boldsymbol{\nabla}$. As $\varphi^*\boldsymbol{\nabla}$ is torsion-free and $\boldsymbol{h}$-compatible, it follows from the fundamental theorem of Riemannian geometry that it must coincide with the connection $\boldsymbol{D}$. In other words, one hast that $\varphi^*\boldsymbol{\nabla} = \boldsymbol{D}$ —as given in equation (2.33). □

A frame formalism on hypersurfaces

In what follows some remarks concerning a frame formalism adapted to hypersurfaces will be made. The hypersurfaces under consideration will be assumed to be either spacelike or timelike. In order to accomodate these two cases, the following index conventions will be made: if the hypersurface is *timelike* so that $\epsilon = 1$, the frame indices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}, \ldots$ take the values $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$; while if the hypersurface is *spacelike* so that $\epsilon = -1$, the indices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}, \ldots$ take the values $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$.

Following the conventions given in the previous paragraph, let $\{\boldsymbol{e}_{\boldsymbol{\alpha}}\} \subset T(\mathcal{S})$ denote a triad of $\boldsymbol{h}$-orthogonal vectors. If $\mathcal{S}$ is spacelike one then has that $\boldsymbol{h}(\boldsymbol{e}_{\boldsymbol{\alpha}}, \boldsymbol{e}_{\boldsymbol{\beta}}) = -\delta_{\boldsymbol{\alpha\beta}}$, while in the timelike case $\boldsymbol{h}(\boldsymbol{e}_{\boldsymbol{\alpha}}, \boldsymbol{e}_{\boldsymbol{\beta}}) = \mathrm{diag}(1,-1,-1)$. Using the push-forward $\varphi_* : T(\mathcal{S}) \to T(\mathcal{M})$ one obtains the vectors $\varphi_*\boldsymbol{e}_{\boldsymbol{\alpha}}$ defined on the restriction of $T(\mathcal{M})$ to $\mathcal{S}$. The triad $\{\boldsymbol{e}_{\boldsymbol{\alpha}}\}$ can be naturally extended to a tetrad $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ on the restriction of $T(\mathcal{M})$ to $\mathcal{S}$ by setting $\boldsymbol{e}_{\mathbf{0}} = \boldsymbol{\nu}^\sharp$ in the spacelike case and $\boldsymbol{e}_{\mathbf{3}} = \boldsymbol{\nu}^\sharp$ in the timelike case.

In order to simplify the presentation, the notation $\boldsymbol{e}_{\boldsymbol{\alpha}}$ will often be used to denote both the vectors of $T(\mathcal{S})$ and their push-forward to $T(\mathcal{M})$. The appropriate point of view should be clear from the context. In the cases where confusion may arise, it is convenient to make use of abstract index notation: given $\boldsymbol{e}_{\boldsymbol{\alpha}} \in T(\mathcal{S})$, we shall write $e_{\boldsymbol{\alpha}}{}^{\alpha}$; its push-forward $\varphi_*\boldsymbol{e}_{\boldsymbol{\alpha}} \in T(\mathcal{M})$ will be denoted by $e_{\boldsymbol{\alpha}}{}^{a}$. Similarly, $\omega^{\boldsymbol{\alpha}} \in T^*(\mathcal{M})$ will often be written as $\omega_{\boldsymbol{\alpha}}{}^{a}$, while the pull-back $\varphi^*\boldsymbol{\omega}^{\boldsymbol{\alpha}}$ will be denoted by $\omega^{\boldsymbol{\alpha}}{}_{\alpha}$. Given $\boldsymbol{u} \in T(\mathcal{S})$, one has that $u^{\boldsymbol{\alpha}} \equiv \langle \varphi^*\boldsymbol{\omega}^{\boldsymbol{\alpha}}, \boldsymbol{u}\rangle = \langle \boldsymbol{\omega}^{\boldsymbol{\alpha}}, \varphi_*\boldsymbol{u}\rangle$. Writen in index notation $u^a\omega_{\boldsymbol{\alpha}}{}^a = u^\alpha \omega^{\boldsymbol{\alpha}}{}_\alpha$. That is, the (spatial) components of $\boldsymbol{u}$ and its push-forward $\varphi_*\boldsymbol{u}$ coincide.

As a consequence of having two covariant derivatives, one also has two sets of directional covariant derivatives. First of all, acting on spacetime objects, $\nabla_{\boldsymbol{a}} = e_{\boldsymbol{a}}{}^a\nabla_a$, so that in particular $\nabla_{\boldsymbol{\alpha}} = e_{\boldsymbol{\alpha}}{}^a\nabla_a$. Secondly, acting on hypersurface-defined objects, one has $D_{\boldsymbol{\alpha}} = e_{\boldsymbol{\alpha}}{}^\alpha D_\alpha$. The connection coefficients of $\boldsymbol{D}$ with respect to $\{\boldsymbol{e}_{\boldsymbol{\alpha}}\}$ are given by $\gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}{}_{\boldsymbol{\gamma}} \equiv \langle \omega^{\boldsymbol{\beta}}, D_{\boldsymbol{\alpha}} e_{\boldsymbol{\gamma}}\rangle$. Now, given $\boldsymbol{u} \in T(\mathcal{S})$ and $\boldsymbol{\alpha} \in T^*(\mathcal{S})$ and defining

$$D_{\boldsymbol{\alpha}} u^{\boldsymbol{\beta}} \equiv \langle \boldsymbol{\omega}^{\boldsymbol{\beta}}, D_{\boldsymbol{\alpha}}\boldsymbol{u}\rangle, \qquad D_{\boldsymbol{\alpha}}\boldsymbol{\alpha}_{\boldsymbol{\beta}} \equiv \langle D_{\boldsymbol{\alpha}}\boldsymbol{\alpha}, \boldsymbol{e}_{\boldsymbol{\beta}}\rangle,$$

one has, by analogy to (2.27) that

$$D_{\boldsymbol{\alpha}} u^{\boldsymbol{\beta}} = \boldsymbol{e}_{\boldsymbol{\alpha}}(u^{\boldsymbol{\beta}}) + \gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}{}_{\boldsymbol{\gamma}} u^{\boldsymbol{\gamma}}, \qquad D_{\boldsymbol{\alpha}}\alpha_{\boldsymbol{\beta}} = e_{\boldsymbol{\alpha}}(\alpha_{\boldsymbol{\beta}}) - \gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\beta}}\alpha_{\boldsymbol{\gamma}}.$$

In order to investigate the directional covariant derivatives $\nabla_{\boldsymbol{\alpha}}$ and $D_{\boldsymbol{\alpha}}$ one makes use of the formula (2.33) with $\boldsymbol{w} = \boldsymbol{e}_{\boldsymbol{\alpha}}$, $\boldsymbol{u} = \boldsymbol{e}_{\boldsymbol{\beta}}$ so that $\varphi_*(D_{\boldsymbol{\alpha}}\boldsymbol{e}_{\boldsymbol{\beta}}) = \nabla_{\boldsymbol{\alpha}}(\varphi_*\boldsymbol{e}_{\boldsymbol{\beta}}) = \nabla_{\boldsymbol{\alpha}}\boldsymbol{e}_{\boldsymbol{\beta}}$ —the last equality given in a slight abuse of notation as $\nabla_{\boldsymbol{\alpha}}$ acts on spacetime objects. From the definition of connection coefficients one has that

$$\begin{aligned}\Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}{}_{\boldsymbol{\gamma}} &= \langle \boldsymbol{\omega}^{\boldsymbol{\beta}}, \nabla_{\boldsymbol{\alpha}}\boldsymbol{e}_{\boldsymbol{\gamma}}\rangle = \langle \boldsymbol{\omega}^{\boldsymbol{\beta}}, \varphi_*(D_{\boldsymbol{\alpha}}\boldsymbol{e}_{\boldsymbol{\gamma}})\rangle \\ &= \langle \varphi^*\omega^{\boldsymbol{\beta}}, D_{\boldsymbol{\alpha}}\boldsymbol{e}_{\boldsymbol{\gamma}}\rangle = \gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}{}_{\boldsymbol{\gamma}}.\end{aligned} \tag{2.34}$$

Given a vector $\boldsymbol{u} \in T(\mathcal{M})$ which is *spatial* (i.e. $u^{\boldsymbol{\nu}} \equiv \nu_a u^a = 0$) and recalling that $\nabla_{\boldsymbol{a}} u^{\boldsymbol{b}} \equiv e_{\boldsymbol{a}}{}^a \omega^{\boldsymbol{b}}{}_b \nabla_a u^b$, using (2.27) one has that

$$\nabla_{\boldsymbol{a}} u^{\boldsymbol{b}} = \boldsymbol{e}_{\boldsymbol{a}}(u^{\boldsymbol{b}}) + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_\gamma u^\gamma. \tag{2.35}$$

Restricting the free frame indices in the above expression and using (2.34) one finds

$$\begin{aligned}\nabla_{\boldsymbol{\alpha}} u^{\boldsymbol{\beta}} &= \boldsymbol{e}_{\boldsymbol{\alpha}}(u^{\boldsymbol{\beta}}) + \Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}{}_\gamma u^\gamma \\ &= \boldsymbol{e}_{\boldsymbol{\alpha}}(u^{\boldsymbol{\beta}}) + \gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}{}_\gamma u^\gamma = D_{\boldsymbol{\alpha}} u^{\boldsymbol{\beta}}.\end{aligned}$$

The intrinsic curvature tensors on the hypersurface

In order to describe the intrinsic curvature of the submanifold $\mathcal{S}$, one considers the ***3-dimensional Riemann curvature tensor*** $r^\gamma{}_{\delta\alpha\beta}$ of the Levi-Civita connection $\boldsymbol{D}$ of the intrinsic metric $\boldsymbol{h}$. Given $v \in T(\mathcal{S})$, and recalling that $\boldsymbol{D}$ is torsion-free, one has by analogy to equation (2.8) that

$$D_\alpha D_\beta v^\gamma - D_\beta D_\gamma v^\gamma = r^\gamma{}_{\delta\alpha\beta} v^\delta.$$

Being the Riemann tensor of a Levi-Civita connection one has the symmetries

$$r_{\gamma\delta\alpha\beta} = r_{[\gamma\delta]\alpha\beta} = r_{\gamma\delta[\alpha\beta]} = r_{[\gamma\delta][\alpha\beta]},$$
$$r_{\gamma\delta\alpha\beta} = r_{\alpha\beta\gamma\delta}, \qquad r_{\gamma[\delta\alpha\beta]} = 0.$$

In what follows, let $r_{\delta\beta} \equiv r^{\gamma}{}_{\delta\gamma\beta}$ and $r \equiv h^{\delta\beta} r_{\delta\beta}$ denote, respectively, the Ricci tensors and scalars of $\boldsymbol{D}$. It is convenient to also consider the trace-free part of the ***3-dimensional Ricci tensor*** $s_{\alpha\beta}$ and the ***3-dimensional Schouten tensor*** $l_{\alpha\beta}$ given by

$$s_{\alpha\beta} \equiv r_{\{\alpha\beta\}} = r_{\alpha\beta} - \tfrac{1}{3} r h_{\alpha\beta}, \qquad l_{\alpha\beta} \equiv s_{\alpha\beta} + \tfrac{1}{12} r h_{\alpha\beta}.$$

The 3-dimensionality of the submanifold $\mathcal{S}$ leads to the decomposition

$$r_{\gamma\delta\alpha\beta} = 2h_{\gamma[\alpha} l_{\beta]\delta} + 2h_{\delta[\beta} l_{\alpha]\gamma}.$$

A computation using the above decompositions shows that the second Bianchi identity takes in in this case the form

$$D^{\alpha} s_{\alpha\beta} = \tfrac{1}{6} D_{\beta} r.$$

Given the $\boldsymbol{h}$-orthogonal triad $\{\boldsymbol{e_\alpha}\}$ and its associated co-frame basis $\{\omega^{\boldsymbol{\alpha}}\}$ one defines the components $r^{\boldsymbol{\gamma}}{}_{\boldsymbol{\delta\alpha\beta}} \equiv e_{\boldsymbol{\alpha}}{}^{\alpha} e_{\boldsymbol{\beta}}{}^{\beta} \omega^{\boldsymbol{\gamma}}{}_{\gamma} e_{\boldsymbol{\delta}}{}^{\delta} r^{\gamma}{}_{\delta\alpha\beta}$. A computation similar to that leading to equation (2.29) yields

$$r^{\boldsymbol{\gamma}}{}_{\boldsymbol{\delta\alpha\beta}} = \boldsymbol{e_\alpha}(\gamma_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\delta}}) - \boldsymbol{e_\beta}(\gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\delta}}) + \gamma_{\boldsymbol{\epsilon}}{}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\delta}}(\gamma_{\boldsymbol{\beta}}{}^{\boldsymbol{\epsilon}}{}_{\boldsymbol{\alpha}} - \gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\epsilon}}{}_{\boldsymbol{\beta}}) + \gamma_{\boldsymbol{\beta}}{}^{\boldsymbol{\epsilon}}{}_{\boldsymbol{\delta}}\gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\epsilon}} - \gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\epsilon}}{}_{\boldsymbol{\delta}}\gamma_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\epsilon}}. \quad (2.36)$$

Moreover, the definition of the torsion tensor implies:

$$[\boldsymbol{e_\alpha}, \boldsymbol{e_\beta}] = (\gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\beta}} - \gamma_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\alpha}})\, \boldsymbol{e_\gamma}.$$

Extrinsic curvature

The discussion in Section 2.7.2 concerning the Weingarten map can be specialised to the case of the tangent space of an hypersurface. This leads to the notion of ***extrinsic curvature*** or ***second fundamental form*** of the hypersurface $\mathcal{S}$. The latter is defined via the map $\boldsymbol{K} : T(\mathcal{S}) \times T(\mathcal{S}) \to \mathbb{R}$ given by

$$\boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v}) \equiv \langle \nabla_{\boldsymbol{u}} \boldsymbol{\nu}, \boldsymbol{v} \rangle = \boldsymbol{g}(\nabla_{\boldsymbol{u}} \boldsymbol{\nu}^{\sharp}, \boldsymbol{v}). \quad (2.37)$$

From the discussion of the Weingarten map it follows that $\boldsymbol{K}$ as defined above is a symmetric 3-dimensional tensor. In abstract index notation the latter will be written as $K_{\alpha\beta}$.

Now, given a orthonormal frame $\{\boldsymbol{e_\alpha}\}$ on $\mathcal{S}$ and choosing $\boldsymbol{v} = \boldsymbol{e_\alpha}$ and

$\boldsymbol{u} = \boldsymbol{e_\beta}$ in the defining formula (2.37) one finds that the components $K_{\boldsymbol{\alpha\beta}}$ of are given by

$$K_{\boldsymbol{\alpha\beta}} = \nabla_{\boldsymbol{\alpha}} n_{\boldsymbol{\beta}} \equiv \langle \nabla_{\boldsymbol{\beta}} \boldsymbol{\nu}, \boldsymbol{e_\alpha} \rangle = \langle \nabla_{\boldsymbol{\beta}} \boldsymbol{\omega^\nu}, \boldsymbol{e_\alpha} \rangle \tag{2.38}$$

so that, comparing with the definition of connection coefficients one finds that

$$\begin{aligned} K_{\boldsymbol{\alpha\beta}} &= \Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{a}}{}_{\boldsymbol{\nu}} \eta_{\boldsymbol{a\beta}} \\ &= -\Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{a}}{}_{\boldsymbol{\beta}} \eta_{\boldsymbol{a\nu}} = -\epsilon \Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\nu}}{}_{\boldsymbol{\beta}}. \end{aligned}$$

Now, looking again at equation (2.35) and setting $\boldsymbol{a} \mapsto \boldsymbol{\alpha}$, $\boldsymbol{b} \mapsto \boldsymbol{\nu}$ one obtains

$$\begin{aligned} \nabla_{\boldsymbol{\alpha}} v^{\boldsymbol{\nu}} &= \boldsymbol{e_\alpha}(v^{\boldsymbol{\nu}}) + \Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\nu}}{}_{\boldsymbol{\gamma}} v^{\boldsymbol{\gamma}} \\ &= \Gamma_{\boldsymbol{\alpha}}{}^{\boldsymbol{\nu}}{}_{\boldsymbol{\gamma}} v^{\boldsymbol{\gamma}} = -\epsilon K_{\boldsymbol{\alpha\gamma}} v^{\boldsymbol{\gamma}}, \end{aligned} \tag{2.39}$$

The Gauss-Codazzi and Codazzi-Mainardi equations

The curvature tensors of the connections $\boldsymbol{\nabla}$ and $\boldsymbol{D}$ are related to each other by means of the ***Gauss-Codazzi equation***

$$R_{\boldsymbol{\alpha\beta\gamma\delta}} = r_{\boldsymbol{\alpha\beta\gamma\delta}} + K_{\boldsymbol{\alpha\gamma}} K_{\boldsymbol{\beta\delta}} - K_{\boldsymbol{\alpha\delta}} K_{\boldsymbol{\beta\gamma}}, \tag{2.40}$$

and the ***Codazzi-Mainardi equation***

$$R_{\boldsymbol{\alpha\nu\beta\gamma}} = D_{\boldsymbol{\beta}} K_{\boldsymbol{\gamma\alpha}} - D_{\boldsymbol{\gamma}} K_{\boldsymbol{\beta\alpha}}. \tag{2.41}$$

The proof of the Gauss-Codazzi equation follows by considering the commutator of $\boldsymbol{\nabla}$, equation (2.8), on the the frame vectors $\boldsymbol{e_\alpha}$:

$$\nabla_a \nabla_b e_{\boldsymbol{\delta}}{}^c - \nabla_b \nabla_a e_{\boldsymbol{\delta}}{}^c = R^c{}_{dab} e_{\boldsymbol{\delta}}{}^d = R^c{}_{\boldsymbol{\delta} ab}.$$

Contracting the previous equation with $e_{\boldsymbol{\alpha}}{}^a e_{\boldsymbol{\beta}}{}^b \omega^{\boldsymbol{\gamma}}{}_c$, and then using

$$\nabla_b e_{\boldsymbol{\delta}}{}^c = \omega^{\boldsymbol{b}}{}_b \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{a}}{}_{\boldsymbol{\delta}} e_{\boldsymbol{a}}{}^c$$

together with formulae (2.34) and (2.39) and the expression for the components of the 3-dimnsional Riemann tensor in terms of the connection coefficients, equation (2.36), yields (2.40). The proof of the Codazzi-Mainardi equation (2.41) involves less computations. In this case one evaluates the commutator of covarinat derivatives on the 1-form $\boldsymbol{\nu}$. Contracting with $e_{\boldsymbol{\alpha}}{}^a e_{\boldsymbol{\beta}}{}^b e_{\boldsymbol{\gamma}}{}^c$ one readily finds that

$$\nabla_{\boldsymbol{\alpha}} \nabla_{\boldsymbol{\beta}} \nu_{\boldsymbol{\gamma}} - \nabla_{\boldsymbol{\beta}} \nabla_{\boldsymbol{\alpha}} \nu_{\boldsymbol{\gamma}} = -R^{\boldsymbol{\nu}}{}_{\boldsymbol{\gamma\alpha\beta}},$$

where $R^{\boldsymbol{\nu}}{}_{\boldsymbol{\gamma\alpha\beta}} \equiv R^d{}_{cab}\nu_d e_{\boldsymbol{\gamma}}{}^c e_{\boldsymbol{\alpha}}{}^a e_{\boldsymbol{\beta}}{}^b$. Now, using equation (2.38) one finds that

$$\nabla_{\boldsymbol{\alpha}} K_{\boldsymbol{\beta\gamma}} - \nabla_{\boldsymbol{\beta}} K_{\boldsymbol{\alpha\gamma}} = -R^{\boldsymbol{\nu}}{}_{\boldsymbol{\gamma\alpha\beta}}.$$

Formula (2.41) readily follows from the above by noticing that $\nabla_{\boldsymbol{\alpha}} K_{\boldsymbol{\beta\gamma}} = D_{\boldsymbol{\alpha}} K_{\boldsymbol{\beta\gamma}}$ as $K_{\boldsymbol{\beta\gamma}}$ corresponds to the spatial components of a spatial tensor.

A remark concerning foliations

The discussion in the previous subsections was restricted to a single hypersurface $\mathcal{S}$. However, it can be readily extended to a foliation $\{\mathcal{S}_t\}$. In this case the contravariant version of the normal $\boldsymbol{\nu}^\sharp$ and the unit vector $\boldsymbol{t}$ generating the congruence coincide. Moreover, one has a distribution which is integrable so that the Weingarten tensor $\boldsymbol{\chi} \in \langle \boldsymbol{t} \rangle^\perp \otimes \langle \boldsymbol{t} \rangle^\perp|_p$, for $p \in \mathcal{M}$ can be identified with the second fundamental form $\boldsymbol{K} \in T_p\mathcal{S}_{t(p)} \otimes T_p\mathcal{S}_{t(p)}$ where $t(p) \in \mathbb{R}$ is the only value of the time function such that $p \in \mathcal{S}_{t(p)}$. In particular one has that $\chi_{\alpha\beta} = \chi_{(\alpha\beta)}$.

2.8 Further reading

There is a vast range of books on Differential Geometry ranging from introductory texts to comprehensive monographs. An introductory discussion geared towards applications in General Relativity can be found in the first chapter of Stewart (1991) or the second and third chapters of Wald (1984). A more extensive introduction with broader applications in Physics is Frankel (2003). A more advanced and terse discussion, again aimed at applications in Physics, is the classical textbook Choquet-Bruhat et al. (1982). A systematic and coherent discussion of the theory from a modern mathematical point of view covering topological manifolds, smooth manifolds and differential geometry can be found in Lee (1997, 2000, 2002). A more concise alternative to the latter three books is given Willmore (1993). A very good monograph on Lorentzian geometry with applications to General Relativity is given in O'Neill (1983). Readers who like the style of this reference will also find the brief summary of Differential Geometry given in the first chapter of O'Neill (1995) useful. The present discussion of Differential Geometry has avoided the use of the language of fibre bundles. Readers interested in the later are referred to Taubes (2011).

Books on numerical Relativity like Baumgarte and Shapiro (2010) and Alcubierre (2008) also provide introductions to the. In these references, the reader will encounter an approach to this topic based on the so-called *pro-*

jection formalism. A more detailed discussion, also aimed at Numerical Relativity, can be found in Gourgoulhon (2012).