Global evaluations of static black holes

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Abstract

The conformal structure of static, spherically symmetric electrovacuum black holes is analysed from the point of view of Cauchy problem in General Relativity. Key tools in this discussion are the extended conformal field equations and generalised conformal Gaussian gauge systems. The implied evolution equations allow to construct global numerical evaluations of the Schwarzschild and Reissner-Nordström spacetimes by solving a system of transport equations. These transport equations are also used to discuss by analytic means the behaviour of the conformal fields along some special conformal curves. Evidence is obtained of the existence of a regular conformal representation of the extreme Reissner-Nordström spacetime.

1 Introduction

The purpose of the present article is to discuss, from the perspective of an initial value problem, certain aspects of the conformal boundary of spherically symmetric solutions, $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu}, \tilde{F}_{\mu\nu})$, to the Einstein-Maxwell field equations with vanishing Cosmological constant

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = \tilde{F}_{\mu\lambda}\tilde{F}^{\lambda}{}_{\nu} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{F}_{\lambda\rho}\tilde{F}^{\lambda\rho}, \tag{1a}$$

$$\tilde{\nabla}^{\mu}\tilde{F}_{\mu\nu} = 0, \tag{1b}$$

$$\tilde{\nabla}_{[\mu}\tilde{F}_{\nu\lambda]} = 0, \tag{1c}$$

where $\tilde{R}_{\mu\nu}$ denotes the Ricci tensor of the Lorentzian metric $\tilde{g}_{\mu\nu}$, and $\tilde{F}_{\mu\nu}$ is the Faraday tensor. As a consequence of the Birkhoff theorem the solutions are described by the Reissner-Nordström family of solutions.

The structure of the the conformal boundary of the maximal analytic extension of the Reissner-Nordström spacetime in general and the Schwarzschild spacetime in particular are well known and can be found in several books —see e.g. [10, 9]. Despite this, to the best of the author's knowledge, the only place in the literature where a discussion of the conformal boundary from the point of view of an initial value problem is the PhD thesis [17] which is the main inspiration of the present analysis —see also [18]. The purpose of the present analysis is to identify conformal structures which could be of potential use in the analysis of the non-linear stability of the Schwarzschild and Reissner-Nordström spacetimes in particular, and the Kerr-Newman spacetime in general.

Notations and conventions

In what follows μ, $\nu, \ldots$ will denote spacetime tensorial indices ranging $0, \ldots, 3$. The indices α, $\beta, \ldots$ are spatial tensorial indices ranging 1, 2, 3. The signature convention for the Lorentzian

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metrics is $(+,-,-,-)$. Thus, the induced metrics on spacelike hypersurfaces are negative definite. The Latin indices i, $j,\ldots$ denote spacetime frame indices, while a, $b,\ldots$ correspond to spatial frame ones. The indices A, $B,\ldots$ denote abstract spinor indices. The spinorial conventions of Penrose & Rindler [14] will be followed throughout.

2 The Schwarzschild and Reissner-Nordström spacetimes

In this section we briefly discuss in an unified fashion some results concerning the Schwarzschild and Reissner-Nordström spacetimes that will be used in the present article.

2.1 Basic expressions and coordinates

The line element and the Faraday tensor of the Reissner-Nordström spacetime is given in standard coordinates $(t,\tilde{r},\theta,\varphi)$ by

$$\tilde{\boldsymbol{g}} = \left(1-\frac{2m}{\tilde{r}}+\frac{q^2}{\tilde{r}^2}\right)\mathbf{d}t\otimes\mathbf{d}t - \left(1-\frac{2m}{\tilde{r}}+\frac{q^2}{\tilde{r}^2}\right)^{-1}\mathbf{d}\tilde{r}\otimes\mathbf{d}\tilde{r}+\tilde{r}^2\left(\mathbf{d}\theta\otimes\mathbf{d}\theta+\sin^2\theta\mathbf{d}\varphi\otimes\mathbf{d}\varphi\right),$$
$$\tilde{\boldsymbol{F}} = \frac{q}{2\tilde{r}^2}\mathbf{d}t\wedge\mathbf{d}\tilde{r}.$$

All throughout it is assumed that

$$m>0, \quad m^2\geq q^2.$$

Isotropic coordinates (t,r,θ,φ) can be introduced via the coordinate transformation

$$r=\frac{1}{2}\left(\tilde{r}-m+\sqrt{\tilde{r}^2-2m\tilde{r}+q^2}\right), \quad \tilde{r}=\frac{1}{4r}(2r+m+q)(2r+m-q),$$

so that one obtains the line element

$$\tilde{\boldsymbol{g}} = \frac{\left(1+\dfrac{q^2-m^2}{4r^2}\right)^2}{\left(1+\dfrac{m+q}{2r}\right)^2\left(1+\dfrac{m-q}{2r}\right)^2}\mathrm{d}t^2 - \left(1+\frac{m+q}{2r}\right)^2\left(1+\frac{m-q}{2r}\right)^2\left(\mathrm{d}r^2+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\mathrm{d}\varphi^2\right). \tag{2}$$

In these coordinates the Faraday tensor takes the form

$$\tilde{\boldsymbol{F}} = \frac{q}{2r^2}\frac{\left(1+\dfrac{q^2-m^2}{4r^2}\right)}{\left(1+\dfrac{m+q}{2r}\right)\left(1+\dfrac{m-q}{2r}\right)}\mathrm{d}t\wedge\mathrm{d}r. \tag{3}$$

When $q=0$ the expression (2) reduces to the well-known line element of the Schwarzschild spacetime in isotropic coordinates:

$$\tilde{\boldsymbol{g}} = \frac{\left(1-\dfrac{m}{2r}\right)^2}{\left(1+\dfrac{m}{2r}\right)^2}\mathrm{d}t^2 - \left(1+\frac{m}{2r}\right)^4\left(\mathrm{d}r^2+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\mathrm{d}\varphi^2\right).$$

2.2 Time symmetric initial data

From the expressions in the previous section one can readily construct time symmetric initial data for the Reissner-Nordström and Schwarzschild spacetimes.

Let $(\tilde{\mathcal{S}},\tilde{h}_{\alpha\beta},\tilde{K}_{\alpha\beta},\tilde{F}_{\alpha\beta})$ denote an initial data set for the Einstein-Maxwell field equations. If $\tilde{\mathcal{S}}$ is regarded as an hypersurface of an spacetime $\tilde{\mathcal{M}}$, then the Faraday can be decomposed in its electric and magnetic parts with respect to the normal to $\tilde{\mathcal{S}}$. In what follows, we will consider time symmetric initial data sets —notice that as the spacetimes under consideration are static, there is such data. For the case of Einstein-Maxwell system, time symmetric initial data sets are characterised by the vanishing of both the extrinsic curvature $\tilde{K}_{\alpha\beta}$ and the magnetic part of $\tilde{F}_{\alpha\beta}$. One can readily verify that the hypersurface $\tilde{\mathcal{S}} = \{t = 0\}$ of the Reissner-Nordström/Schwarzschild spacetime is time symmetric —the spacetime has a discrete time reflexion symmetric with respect to this surface. From expressions (2) and (3) it follows that the intrinsic metric $\tilde{h}_{\alpha\beta}$ of this hypersurface and the corresponding electric part of the Faraday tensor are given by

$$\tilde{\boldsymbol{h}} = -\phi^2\chi^2\boldsymbol{\delta}, \tag{4a}$$

$$\tilde{\boldsymbol{E}} = \frac{q}{r^2}\phi^{-1}\chi^{-1}\mathrm{d}r, \tag{4b}$$

with

$$\phi \equiv \left(1 + \frac{m+q}{2r}\right), \quad \chi \equiv \left(1 + \frac{m-q}{2r}\right).$$

and

$$\boldsymbol{\delta} = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2$$

the 3-dimensional flat metric —hence, the initial metric is conformally flat. If $m > q$ the 3-metric (4a) has two asymptotically Euclidean ends. In the extremal case, $q = m$, one has one asymptotically Euclidean end and one trumpet-like end. The tensors $\tilde{h}_{\alpha\beta}$ and $\tilde{E}_\alpha$ as defined by (4a)-(4b) satisfy the time symmetric constraints

$$\tilde{r} = 2\tilde{E}_\alpha\tilde{E}^\alpha,$$
$$\tilde{D}_\alpha\tilde{E}^\alpha = 0.$$

3 Spinorial structures on the 3-sphere

The spacetimes to be analysed in the present article will be regarded as having spatial sections which can be compactified to $\mathbb{S}^3$. In order to exploit the spherical symmetry of the setting, one would like to introduce a frame formalism which makes this symmetry manifest. Frames of this form are not globally defined on $\mathbb{S}^3$, thus one considers the conformal field equations on a bundle manifold which is a subset of the bundle space-spinors over $\mathbb{S}^3$. This approach is inspired by the discussion in [5] —see also [2].

Let $\boldsymbol{\hbar}$ denote the (negative definite) standard metric of $\mathbb{S}^3$ in the spherical coordinates (ψ, θ, φ) with $0 \leq \psi \leq \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. One has that

$$\boldsymbol{\hbar} = -\left(\mathrm{d}\psi^2 + \sin^2\psi\mathrm{d}\theta^2 + \sin^2\psi\sin^2\theta\mathrm{d}\varphi^2\right). \tag{5}$$

In what follows, the *North Pole* ($\psi = 0$) will be denoted by i_1 while the *South Pole* ($\psi = \pi$) will be denoted by i_2.

3.1 The bundle manifold $\mathcal{C}(\mathbb{S}^3)$

As mentioned previously, instead of working with $\mathbb{S}^3$, an associated bundle manifold $\mathcal{C}(\mathbb{S}^3)$ will be considered. Let $SU(\mathbb{S}^3)$ denote the bundle of normalised spin frames over $\mathbb{S}^3$ with structure group $SU(2,\mathbb{C})$ and projection π onto $\mathbb{S}^3$. It is convenient to think of $\mathbb{S}^3$ as a spacelike hypersurface with future directed (spinorial) normal $\tau_{AA'}$ of a 4-dimensional Lorentzian manifold $(\mathcal{M}, g)$ and of the group $SU(2,\mathbb{C})$ as

$$SU(2,\mathbb{C}) = \{t^A{}_B \in SL(2,\mathbb{C}) \,|\, \tau_{AA'}t^A{}_B\bar{t}^{A'}{}_{B'} = \tau_{BB'}\}.$$

3.1.1 Construction

To obtain the manifold $\mathcal{C}(\mathbb{S}^3)$ one starts by choosing a fixed spin dyad $\delta^*{}_A$ at i_1. Any other spin frame is of the form $\delta_A(t) = \delta^*{}_B t^B{}_A$, $t = (t^B{}_A) \in SU(2,\mathbb{C})$. In particular, for some values of t, the frame vector $e_3(t) = \sigma_3{}^{AB}\delta_A(t)\delta_B(t)$ corresponds to the radial vector at i_1. Keeping t fixed, one then constructs the $\boldsymbol{\hbar}$-geodesic starting at i_1 that has tangent vector $e_3(t)$ and that ends at i_2. The coordinate ψ is an affine parameter of this geodesic that vanishes at i_1. The spin dyad δ_A is then propagated along the geodesic. For a particular value of ψ the spin dyad so constructed will be denoted by $\delta_A(\psi,t)$. One then sets

$$\mathcal{C}(\mathbb{S}^3) = \left\{ \delta_A(\psi,t) \in SU(\mathbb{S}^3) \,|\, 0 \leq \psi \leq \pi \right\}.$$

The bundle manifold $\mathcal{C}(\mathbb{S}^3)$ consists of two disconnected components $\mathcal{I}_1^0 = \{\psi = 0\}$ and $\mathcal{I}_2^0 = \{\psi = \pi\}$. One has that $\mathcal{I}_1^0 \simeq \mathcal{I}_2^0 \simeq SU(2,\mathbb{C})$, so that the components of the boundary can be regarded as the blow up of the points at infinity i_1 and i_2.

3.1.2 Lifts to $\mathcal{C}(\mathbb{S}^3)$

Any smooth spinor field on $\mathbb{S}^3$ is represented on $\mathcal{C}(\mathbb{S}^3)$ by a *spinor valued function* given at $\delta_A \in \mathcal{C}(\mathbb{S}^3)$ by the components of the spinor in the dyad defined by δ_A. This procedure will be referred to as the *lift* of the spinor field. The lift to $\mathcal{C}(\mathbb{S}^3)$ of any symmetric valence 2 spinor on $\mathbb{S}^3$ can be spanned in terms of symmetric spinors x_{AB}, y_{AB}, z_{AB} such that

$$x_A{}^Q x_{BQ} = \tfrac{1}{2}\epsilon_{AB}, \quad x_{BQ}y_A{}^Q = \tfrac{1}{\sqrt{2}}y_{AB}, \quad x_{BQ}z_A{}^Q = -\tfrac{1}{\sqrt{2}}z_{AB}, \quad y_A{}^Q z_{BQ} = -\tfrac{1}{2\sqrt{2}}x_{AB} + \tfrac{1}{4}\epsilon_{AB},$$

and $y_A{}^Q y_{BQ} = z_A{}^Q z_{BQ} = 0$. Higher valence spinors can be spanned by suitable combinations of these spinors and ϵ_{AB} —[5, 7].

3.1.3 Vector fields on $\mathcal{C}(\mathbb{S}^3)$

The manifold $\mathcal{C}(\mathbb{S}^3)$ has a dimension more than $\mathbb{S}^3$. This extra dimension corresponds to the action of the subgroup $U(1)$ of $SU(2,\mathbb{C})$. In what follows, we will use $t^A{}_B \in SU(2,\mathbb{C})$ and ψ as coordinates on $\mathcal{C}(\mathbb{S}^3)$. In order to obtain a calculus on $\mathcal{C}(\mathbb{S}^3)$, one considers a basis $\{\boldsymbol{X}_+, \boldsymbol{X}_-, \boldsymbol{X}\}$ of the Lie algebra $\mathfrak{su}(2,\mathbb{C})$, such that $\boldsymbol{X}$ is the generator of $U(1)$ and one has the commutation relations

$$[\boldsymbol{X}, \boldsymbol{X}_+] = 2\boldsymbol{X}_+, \quad [\boldsymbol{X}, \boldsymbol{X}_-] = -2\boldsymbol{X}_-, \quad [\boldsymbol{X}_+, \boldsymbol{X}_-] = -\boldsymbol{X},$$

and $\boldsymbol{X}_+$ and $\boldsymbol{X}_-$ are complex conjugates of each other. These vector fields are extended to $\mathcal{C}(\mathbb{S}^3)$ by the requirements

$$[\boldsymbol{\partial}_\psi, \boldsymbol{X}] = 0, \quad [\boldsymbol{\partial}_\psi, \boldsymbol{X}_+] = 0, \quad [\boldsymbol{\partial}_\psi, \boldsymbol{X}_-] = 0.$$

The vector fields $\boldsymbol{\partial}_\psi$, $\boldsymbol{X}$, $\boldsymbol{X}_\pm$ constitute a frame field on $\mathcal{C}(\mathbb{S}^3)$. A function f is said to have spin weight s if $\boldsymbol{X}f = 2sf$, with s an integer. Any spinor-valued function on $\mathcal{C}(\mathbb{S}^3)$ has a well defined spin weight —see [5]. To complete the discussion, one requires to consider forms $\boldsymbol{\alpha}^+$, $\boldsymbol{\alpha}^-$ and $\boldsymbol{\alpha}$ which annihilate the the vector fields ∂_τ and ∂_ψ and have the non-vanishing pairings

$$\langle \boldsymbol{\alpha}^+, \boldsymbol{X}_+ \rangle = \langle \boldsymbol{\alpha}^-, \boldsymbol{X}_- \rangle = \langle \boldsymbol{\alpha}, \boldsymbol{X} \rangle = 1.$$

The normalisation conventions being used are such that $2(\boldsymbol{\alpha}^+ \otimes \boldsymbol{\alpha}^- + \boldsymbol{\alpha}^- \otimes \boldsymbol{\alpha}^+)$ pulls back to the standard metric on $\mathbb{S}^2$.

3.1.4 Frame fields, solder forms and connection forms

The vector fields and 1-forms on $\mathcal{C}(\mathbb{S}^3)$ introduced in the previous subsection will be used to span the following frame fields and corresponding solder forms:

$$\boldsymbol{e}_{AB} = x_{AB}\partial_\psi + \csc\psi\, z_{AB}\boldsymbol{X}_+ + \csc\psi\, y_{AB}\boldsymbol{X}_-, \tag{6a}$$
$$\boldsymbol{\sigma}^{AB} = -x^{AB}\mathrm{d}\psi - 2\sin\psi\, y^{AB}\boldsymbol{\alpha}^+ - 2\sin\psi\, z^{AB}\boldsymbol{\alpha}^-. \tag{6b}$$

On has that

$$\boldsymbol{\hbar} = h_{ABCD}\sigma^{AB}\otimes\sigma^{CD}, \quad \langle \boldsymbol{\sigma}^{AB}, \boldsymbol{e}_{CD}\rangle = h^{AB}{}_{CD},$$

where $h_{ABCD} = -\epsilon_{A(C}\epsilon_{D)B}$ is the spinorial counterpart of $-\delta_{ab}$. The associated connection coefficients γ_{ABCD} an be computed using the spinorial version of the Cartan structure equations and that $\mathbb{S}^3$ has constant curvature with Ricci scalar given, in the present conventions, by $r=-6$. One has that

$$\gamma_{ABCD} = \tfrac{1}{2}\cot\psi(\epsilon_{AC}x_{BD} + \epsilon_{BD}x_{AC}). \tag{7}$$

Covariant differentiation on $\mathcal{C}(\mathbb{S}^3)$ is performed using the standard rules. Let F denote the lift to $\mathcal{C}(\mathbb{S}^3)$ of a smooth function, f, on $\mathbb{S}^3$. The covariant derivative $D_{AB}f$ is represented on $\mathcal{C}(\mathbb{S}^3)$ by $\boldsymbol{e}_{AB}F$. In order to ease the notation, in what follows the same symbol will be used to denote a function on $\mathbb{S}^3$ and its lift to $\mathcal{C}(\mathbb{S}^3)$ and write $D_{AB}f = \boldsymbol{e}_{AB}f$. Using this convention, let μ_{AB} denote the lift to $\mathcal{C}(\mathbb{S}^3)$ of the spinorial field μ_{AB} on $\mathbb{S}^3$. The lift of the covariant derivative $D_{AB}\mu_{CD}$ is then given by

$$D_{AB}\mu_{CD} = \boldsymbol{e}_{AB}\mu_{CD} - \gamma_{AB}{}^{P}{}_{C}\mu_{PD} - \gamma_{AB}{}^{P}{}_{D}\mu_{CP}.$$

Similar expressions hold for higher valence spinors.

4 The extended conformal evolution equations for electrovacuum spacetimes

The conformal Einstein-Maxwell field equations have been first discussed in [3]. This formulation of the conformal equations considered connections which are the Levi-Civita connection of some metric. A formulation of the conformal Einstein-Maxwell field equations which allows for the use of Weyl connections has been given in [13]. The latter system equations will be referred to as the *extended conformal field equations*. The (extended) conformal equations allow to discuss the solutions to Einstein-Maxwell field equations (1a)-(1c) in terms of an *unphysical metric* $g_{\mu\nu}$ and and *unphysical Faraday tensor* $F_{\mu\nu}$ related to the *physical metric* $\tilde{g}_{\mu\nu}$ and *physical Faraday tensor* $\tilde{F}_{\mu\nu}$ according to the conformal rescaling

$$g_{\mu\nu} = \Theta^2\tilde{g}_{\mu\nu}, \quad F_{\mu\nu} = \Theta^{-1}\tilde{F}_{\mu\nu} \tag{8}$$

for some (non-negative) conformal factor Θ. In what follows, let $\tilde{\nabla}$, ∇ denote, respectively, the Levi-Civita connections of the metrics $\tilde{g}_{\mu\nu}$, $g_{\mu\nu}$.

4.1 The fields of the extended conformal field equations

Given a metric $\tilde{g}_{\mu\nu}$, its *conformal class* $[\tilde{g}]$ is defined as the set of metric conformally related to $\tilde{g}_{\mu\nu}$. A Weyl connection $\hat{\nabla}$ is the defined as a torsion free connection (not necessarily Levi-Civita) which respects the conformal structure of the conformal class of $\tilde{g}$ in the sense that

$$\hat{\nabla}_\lambda\tilde{g}_{\mu\nu} = -2b_\lambda\tilde{g}_{\mu\nu}, \quad \hat{\nabla}_\lambda g_{\mu\nu} = -2f_\lambda g_{\mu\nu},$$

for some 1-forms b_μ and f_μ. As a consequence of the rescaling (8), it follows that

$$b_\mu = \Theta^{-1}\nabla_\mu\Theta + f_\mu \tag{9}$$

—see e.g. [4, 13]. In what follows, it will be convenient to consider a 1-form d_μ defined via

$$d_\mu \equiv \Theta b_\mu. \tag{10}$$

The extended conformal Einstein-Maxwell field equations discussed in [13] are equations relating the conformal factor Θ, the 1-form d_μ, a g-orthonormal frame e_i, the 1-form f_μ, the connection coefficients $\hat{\Gamma}_i{}^j{}_k$ of the Weyl connection $\hat{\nabla}$, the components of the Schouten tensor $\hat{P}_{ij}$ of $\hat{\nabla}$ with respect to e_i, the components of the rescaled Weyl tensor $d^i{}_{jkl}$ and those of the rescaled Faraday tensor F_{ij} and its derivative $\psi_{ijk} \equiv \hat{\nabla}_i F_{jk}$. In this formulation, the fields Θ and d_μ are not subject to differential equations, and can be fixed through gauge conditions. In the sequel a spinorial version of the equations will be considered. The spinorial counterparts of the fields

$$e_i, \quad f_i, \quad \hat{\Gamma}_i{}^j{}_k, \quad \hat{P}_{ij}, \quad d^i_{jkl}, \quad F_{ij}, \quad \psi_{ijk}$$

will be denoted, respectively, by

$$e_{AA'}, \quad f_{AA'}, \quad \hat{\Gamma}_{AA'BC}, \quad \hat{P}_{AA'BB'}, \quad \phi_{ABCD}, \quad \varphi_{AB}, \quad \psi_{AA'BC}.$$

It this transcription to spinors, the symmetries of the various fields have been exploited. In particular,

$$\hat{\Gamma}_{AA'BC} = \hat{\Gamma}_{AA'(BC)}, \quad \phi_{ABCD} = \phi_{(ABCD)}, \quad \varphi_{AB} = \varphi_{(AB)}, \quad \psi_{AA'BC} = \psi_{AA'(BC)}.$$

4.2 Gauge considerations

In order to extract a system of evolution equations from the extended conformal Einstein-Maxwell equations, a gauge based on certain conformal curves will be employed —see [13].

4.2.1 Generalised conformal Gaussian systems

In what follows let $\hat{\nabla}$ be a Weyl connection obtained from the unphysical Levi-Civita connection ∇ using a 1-form f_μ. In the case of an electrovacuum spacetime, the conformal curves introduced in [13] are characterised by the conditions

$$\dot{x}^\mu \hat{\nabla}_\mu \dot{x}^\nu = 0, \quad \hat{P}_{\mu\nu}\dot{x}^\nu = \tfrac{1}{2}\Theta^2\left(F_{\mu\lambda}F^\lambda{}_\nu - \tfrac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}\right). \tag{11}$$

In the case of vacuum, these conditions reduce to the conformal geodesic equations of [4, 6]. A generalised conformal Gaussian system will be constructed using a congruence of conformal curves. To this end, the g-orthonormal frame e_i is propagated using the conditions

$$\dot{x}^\mu \hat{\nabla}_\mu e_i{}^\nu = 0, \quad e_0{}^\mu = \dot{x}^\mu. \tag{12}$$

The affine parameter of the conformal curves, τ will be used a time coordinate. Spatial coordinates are extended off an initial hypersurface by requiring them to remain constant along a given conformal curve. In [13] it has been shown that a generalised conformal Gaussian system satisfies in addition to (11)-(12), the condition

$$\hat{\Gamma}_0{}^j{}_k = 0, \quad \dot{x}^\mu f_\mu = 0. \tag{13}$$

The spinorial version of the gauge conditions (11)-(13) is given by

$$\tau^{AA'}\hat{\Gamma}_{AA'BC} = 0, \quad \tau^{AA'} f_{AA'} = 0, \quad \tau^{AA'}\hat{P}_{AA'BB'} = \tfrac{1}{2}\Theta^2 \tau^{BB'}\phi_{AB}\bar{\phi}_{A'B'},$$

where $\tau^{AA'}$ is the spinorial counterpart of $\dot{x}^\mu$. It satisfies the normalisation $\tau_{AA'}\tau^{AA'} = 2$.

4.2.2 The *a priori* conformal boundary

The use of a conformal Gaussian gauge system allows an *a priori* identification of the conformal boundary of the spacetime. In [4, 5, 11, 12, 13] it has been shown that if on the initial hypersurface one sets

$$\langle \boldsymbol{n}, \dot{x}\rangle_* = 1, \quad \Theta_* = \kappa^{-1}\Omega, \quad \dot{\Theta}_* = \Theta_*\langle \boldsymbol{b}, \dot{x}\rangle_*, \quad \boldsymbol{b}_* = \Omega^{-1}\mathbf{d}\Omega, \tag{14}$$

where at this stage Ω will be regarded as a non-negative scalar field on the initial hypersurface, κ a further smooth function expressing the remaining conformal freedom in this gauge, and $\boldsymbol{n}$ is the unit normal to the initial hypersurface, then one obtains the following explicit expression for the spacetime conformal factor Θ:

$$\Theta = \kappa^{-1}\Omega\left(1 + \langle \boldsymbol{b}, \dot{x}\rangle_* \tau + \left(\tfrac{1}{4}\langle \boldsymbol{b}, \dot{x}\rangle_*^2 - \frac{\kappa^2}{\omega^2}\right)\tau^2\right), \quad \omega \equiv \frac{2\Omega}{\sqrt{|\boldsymbol{h}^\sharp(\mathbf{d}\Omega, \mathbf{d}\Omega)|}}, \tag{15}$$

where the subscript $_*$ indicates that the relevant quantity is constant along a given conformal geodesic —its actual value defined at $\tau = 0$. Furthermore, the components of the form d_μ —see equation (10)— with respect to a Weyl propagated frame e_k satisfying (12) are given by

$$d_k \equiv \langle \boldsymbol{d}, \boldsymbol{e}_k\rangle = (\dot{\Theta}, \Theta b_{a*}).$$

For convenience, the *a priori conformal boundary* is defined as the set of points for which $\Theta = 0$. As it will be seen in the sequel, points in the *a priori* conformal boundary may not be realised in the actual development of an initial data set of the electrovacuum Einstein field equations.

As pointed out in [4, 11], in a conformal Gaussian system the relation (9) provides information about the nature of the conformal boundary if $f_k \equiv f_\mu e_k{}^\mu$ and $\boldsymbol{e}_k$ remain smooth at the points for which $\Theta = 0$ and $\mathbf{d}\Theta \neq 0$ one has that

$$\boldsymbol{g}^\sharp(\mathbf{d}\Theta, \mathbf{d}\Theta) = 0.$$

Thus, the part of the *a priori* conformal boundary for which $\mathbf{d}\Theta \neq 0$ corresponds to a null hypersurface. The various parts of this null hypersurface are separated by caustic sets. Caustics on the conformal boundary occur whenever $\Theta = 0$ and $\mathrm{d}\Theta = 0$.

In the sequel, points for which $\kappa^{-1}\Omega = 0$ will correspond to the point at infinity of an asymptotically Euclidean end. Clearly, for these points one has $\Theta = 0$. A connected set of points for which $\kappa^{-1}\Omega = 0$ and $|\tau| < \tau_{critical}$ where $\tau_{critical}$ corresponds to the solutions to

$$\left(1 + \langle \boldsymbol{b}, \dot{x}\rangle_* \tau + \left(\tfrac{1}{4}\langle \boldsymbol{b}, \dot{x}\rangle_*^2 - \frac{\kappa^2}{\omega^2}\right)\tau^2\right) = 0,$$

will be called —following the terminology of [5]— a *cylinder at spatial infinity.* The sets for which $\tau = \tau_{critical}$ will be called *critical sets*, and will be interpreted as the sets where null infinity meets spatial infinity. It can be readily checked that the critical sets are caustic sets as $\mathbf{d}\Theta = 0$.

Another possibility for a caustic set occurs when at points for which $\Theta = 0$ and $\mathbf{d}\Omega = 0$. It can be readily checked that if this is the case then one indeed has $\mathbf{d}\Theta = 0$. As discussed in [11], such points correspond to potential "Minkowski-like" timelike infinities. Using the formula of the conformal factor one finds that the time location of the caustic is given by

$$\tau_{i^+} = -2/\langle \boldsymbol{b}, \dot{x}\rangle_*.$$

In particular, if $\langle \boldsymbol{b}, \dot{x}\rangle_* = 0$, one has a conformal representation for which the potential timelike infinity has no finite coordinate location.

Finally, it is noticed that the set of points for which $\Theta = 0$ and $\mathbf{d}\Theta \neq 0$ can be grouped in two categories: (i) those points corresponding to the end points of physical null geodesics, so that these points can be considered as belonging to a *null infinity* in the spirit of the definition of *asymptotic simplicity* —see e.g. [15]; (ii) those points which do not correspond to the end points of null geodesics. In the sequel, the latter category of points will be ascribed to a *Cauchy horizon.* Points in the conformal boundary with $\mathbf{d}\Theta \neq 0$ close to a critical point can be shown to actually belong to a null infinity in the sense described in this paragraph. By continuity the whole connected component of the conformal boundary will also be a null infinity.

Remark. The discussion in the previous paragraphs is only meaningful is the solutions to the conformal evolution equations are regular at the points under discussion.

4.3 Initial data

Initial data for the conformal evolution equations (21a)-(21d) it to be obtained from a suitable solution to the conformal constraint equations. Given a solution $(h_{\alpha\beta}, \chi_{\alpha\beta}, \Omega, \Sigma, E_\alpha, B_\alpha)$ to the *conformal Hamiltonian* and *momentum constraints*

$$2\Omega D_\alpha D^\alpha \Omega - 3D_\alpha \Omega D^\alpha \Omega + \tfrac{1}{2}\Omega^2 r - 3\Sigma^2$$
$$-\tfrac{1}{2}\Omega^2 \left(\chi^2 - \chi_{\alpha\beta}\chi^{\alpha\beta}\right) + 2\Omega\Sigma\chi = \Omega^4 \left(E_\alpha E^\alpha + B_\alpha B^\alpha\right), \tag{16a}$$
$$\Omega^3 D^\alpha(\Omega^{-2}\chi_{\alpha\beta}) - \Omega\left(D_\beta\chi - 2\Omega^{-1}D_\beta\Sigma\right) = \Omega^3 \epsilon_\alpha{}^{\beta\gamma} E_\beta B_\gamma, \tag{16b}$$

and the conformal electromagnetic constraints

$$D^\alpha E_\alpha = 0, \quad D^\alpha B_\alpha = 0, \tag{17}$$

there is a definite procedure for the construction of a solution to the conformal constraint equations —see e.g. [4, 16]. In the previous expressions $h_{\alpha\beta}$ denotes a (negative definite) 3-metric, $\chi_{\alpha\beta}$ is a symmetric tensor corresponding to the extrinsic curvature of the initial unphysical manifold as a hypersurface in the unphysical spacetime, Ω is a conformal factor relating the 3-metric $h_{\alpha\beta}$ to a *physical* 3-metric $\tilde{h}_{\alpha\beta}$, Σ denotes the derivative of the spacetime conformal factor in the direction of the g-normal of the initial hypersurface and E_α and B_α denote two 3-vectors corresponding to the initial values of the electric and magnetic fields. The explicit expressions relating the unphysical data $(h_{\alpha\beta}, \chi_{\alpha\beta}, \Omega, \Sigma, E_\alpha, B_\alpha)$ to the physical data $(\tilde{h}_\alpha, \tilde{\chi}_{\alpha\beta}, \tilde{E}_\alpha, \tilde{B}_\alpha)$ is given by

$$h_{\alpha\beta} = \Omega^2\tilde{h}_{\alpha\beta}, \quad \chi_{\alpha\beta} = \Omega(\tilde{\chi}_{\alpha\beta} + \Sigma\tilde{h}_{\alpha\beta}), \quad E_\alpha = \Omega^{-1}\tilde{E}_\alpha, \quad B_\alpha = \Omega^{-1}\tilde{B}_\alpha. \tag{18}$$

4.4 The manifold $\mathcal{M}_\kappa$

The unphysical spacetime metric $\boldsymbol{g}$ has the spinorial representation

$$\boldsymbol{g} = \epsilon_{AB}\bar{\epsilon}_{A'B'}\boldsymbol{\sigma}^{AA'} \otimes \boldsymbol{\sigma}^{BB'},$$

where $\boldsymbol{\sigma}^{AA'}$ is the *spacetime solder form*. All throughout it will be assumed that $\mathcal{M}$ is threaded by a congruence of conformal curves with parameter τ with tangent vector given by τ^μ. The spinor $\tau^{AA'}$ used in Section 3 is then related to the solder form $\boldsymbol{\sigma}^{AA'}$ via

$$\langle \boldsymbol{\sigma}^{AA'}, \boldsymbol{\partial}_\tau \rangle = \tfrac{1}{\sqrt{2}}\tau^{AA'}.$$

Consistent with the discussion of Section 3, instead of considering the extended conformal field equations on an unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ with the topology of $\mathbb{R}\times\mathbb{S}^3$, the field equations will be lifted to a bundle manifold. To this end, given a non-negative function smooth function, κ, we define

$$\mathcal{C}_\kappa(\mathbb{S}^3) \equiv \left\{\kappa^{1/2}\delta_A \,|\, \delta_A \in \mathcal{C}(\mathbb{S}^3)\right\}.$$

Using the extended bundle manifold $\mathcal{C}_\kappa(\mathbb{S}^3)$, one further defines

$$\mathcal{M}_\kappa \equiv \left\{(\tau, \psi, t^A{}_B) \,|\, \tau \in [-\tau_{max}, \tau_{max}] \subset \mathbb{R},\ (\psi, t^A{}_B) \in \mathcal{C}_\kappa(\mathbb{S}^3)\right\},$$

for some $\tau_{max} > 0$. Following the discussion of [5], it can be seen that $\mathcal{M}_\kappa$ is, in fact, a subbundle of

$$CSL(\mathcal{M}) = \left\{\lambda\delta_A \,|\, \delta_A \in SL(\mathcal{M}),\ \lambda \in \mathbb{R}^+\right\},$$

where $SL(\mathcal{M})$ denotes the bundle of normalised spin frames over $\mathcal{M}$ with structure group given by $SL(2,\mathbb{C})$. In particular, $\mathcal{C}_\kappa(\mathbb{S}^3)$ can be regarded as a submanifold of the extended bundle $CSL(\mathcal{M})$. Furthermore, it can be shown that $\tau^{AA'}$, $\sigma^{AA'}$ and the connection form associated to the Weyl connection $\hat{\nabla}$ can be lifted to $\mathcal{M}_\kappa$. The same procedure can also be applied to all the conformal fields appearing in the extended conformal field equations. The lifts of all these quantities will be denoted with the same symbols.

The manifold $\mathcal{C}_\kappa(\mathbb{S}^3)$ is diffeomorphic to $\mathcal{C}(\mathbb{S}^3)$, thus the coordinates ψ and $t^A{}_B$, as well as the vector fields $\boldsymbol{\partial}_\psi$, $\boldsymbol{X}$, $\boldsymbol{X}_+$ and $\boldsymbol{X}_+$ defined on $\mathcal{C}(\mathbb{S}^3)$ can be carried over in a natural way to $\mathcal{C}_\kappa(\mathbb{S}^3)$. Furthermore, the coordinates can be extended to the whole of $\mathcal{M}_\kappa$ by requiring them to remain constant along a given conformal curve. In the case of the vector fields, the extension is obtaining by demanding that their commutator with $\boldsymbol{\partial}_\tau$ vanishes.

4.5 Some expressions concerning the space spinor decomposition

The frame fields $\boldsymbol{e}_{AA'}$ are related to the frame forms via

$$\langle \boldsymbol{\sigma}^{AA'}, \boldsymbol{e}_{BB'} \rangle = \epsilon_B{}^A \bar{\epsilon}_{B'}{}^{A'}.$$

In terms of the vector fields $\boldsymbol{\partial}_\tau$, $\boldsymbol{\partial}_\psi$, $\boldsymbol{X}_\pm$ one has that

$$\begin{aligned}
\boldsymbol{e}_{AA'} &= e^0_{AA'}\boldsymbol{\partial}_\tau + e^1_{AA'}\boldsymbol{\partial}_\psi + e^+_{AA'}\boldsymbol{X}_+ + e^-_{AA'}\boldsymbol{X}_-,\\
\boldsymbol{\sigma}^{AA'} &= \sigma_0^{AA'}\mathbf{d}\tau + \sigma_1^{AA'}\mathbf{d}\psi + \sigma_+^{AA'}\boldsymbol{\alpha}^+ + \sigma_-^{AA'}\boldsymbol{\alpha}^-.
\end{aligned}$$

The space spinor decomposition of $\boldsymbol{e}_{AA'}$ and $\boldsymbol{\sigma}^{AA'}$ is given by

$$\begin{aligned}
\boldsymbol{e}_{AA'} &= \tfrac{1}{\sqrt{2}}\tau_{AA'}\boldsymbol{\partial}_\tau - \tau^Q{}_{A'}\boldsymbol{e}_{AQ},\\
\boldsymbol{\sigma}^{AA'} &= \tfrac{1}{2}\tau^{AA'}(\tau_{PP'}\boldsymbol{\sigma}^{PP'}) + \tau_Q{}^{A'}\boldsymbol{\sigma}^{QA},
\end{aligned}$$

with

$$\begin{aligned}
\boldsymbol{e}_{AB} &\equiv \tau_{(A}{}^{B'}\boldsymbol{e}_{B)B'} = e^0_{AB}\boldsymbol{\partial}_\tau + e^1_{AB}\boldsymbol{\partial}_\psi + e^+_{AB}\boldsymbol{X}_+ + e^-\boldsymbol{X}_-,\\
\boldsymbol{\sigma}^{AB} &\equiv -\tau^{(A}{}_{P'}\boldsymbol{\sigma}^{B)P'} = \sigma_1^{AB}\mathbf{d}\psi + \sigma_+{}^{AB}\boldsymbol{\alpha}^+ + \sigma_-{}^{AB}\boldsymbol{\alpha}^-,
\end{aligned}$$

and

$$\langle \tau_{AA'}\boldsymbol{\sigma}^{AA'}, \boldsymbol{\partial}_\tau \rangle = \sqrt{2}, \quad \langle \tau_{AA'}\boldsymbol{\sigma}^{AA'}, \boldsymbol{e}_{BC} \rangle = 0, \quad \langle \boldsymbol{\sigma}^{AB}, \boldsymbol{e}_{CD} \rangle = h^{AB}{}_{CD}.$$

Finally, we recall the space spinor split of the unphysical metric $\boldsymbol{g}$:

$$\boldsymbol{g} = \tfrac{1}{2}(\tau_{PP'}\boldsymbol{\sigma}^{PP'}) \otimes (\tau_{QQ'}\boldsymbol{\sigma}^{QQ'}) + h_{ABCD}\boldsymbol{\sigma}^{AB} \otimes \boldsymbol{\sigma}^{CD}. \tag{19}$$

4.6 Structural properties of the conformal evolution equations

The procedure of hyperbolic reduction for the extended conformal Einstein-Maxwell field equations under the assumption of a generalised conformal Gaussian system has been discussed with some detail in [13]. It makes use of a space-spinor formalism based on the spinor $\tau^{AA'}$ tangent to the conformal curves. By means of this formalism, spinors with primed indices can be translated to spinors with only unprimed indices which, in turn, can be readily decomposed in terms of their irreducible decomposition. Thus, instead of dealing with the spinors $e_{AA'}{}^\mu$, $\hat{\Gamma}_{AA'CD}$, $f_{AA'}$ and $\hat{P}_{AA'CC'}$, one considers their space-spinor counterparts $e_{AB}{}^\mu$, $\hat{\Gamma}_{ABCD}$, f_{AB} and $\hat{P}_{ABCD}$. As discussed in [3] readily allows the identification of constraints and hyperbolic evolution equations.

The detailed evolution equations —computed using the suited `xAct` for `Mathematica` will not be required here. Instead, a schematic discussion of their structure will be provided. To this end, let

$$\boldsymbol{\upsilon} \equiv \left(e^{\bar{s}}_{AB}, \hat{\Gamma}_{ABCD}, \hat{P}_{ABCD}\right), \quad \boldsymbol{\phi} \equiv (\phi_{ABCD}), \quad \boldsymbol{\varphi} \equiv (\phi_{AB}), \quad \boldsymbol{\psi} \equiv (\psi_{ABCD}), \tag{20}$$

where it is understood that $\boldsymbol{\upsilon}$, $\boldsymbol{\phi}$, $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ contain only the independent irreducible components of the respective spinors. In terms of these quantities the conformal Einstein-Maxwell propagation equations can be written as:

$$\partial_\tau\boldsymbol{\upsilon} = \mathbf{K}\boldsymbol{\upsilon} + \mathbf{Q}(\boldsymbol{\upsilon},\boldsymbol{\upsilon}) + \mathbf{R}(\boldsymbol{\varphi},\boldsymbol{\psi}) + \mathbf{T}(\boldsymbol{\phi},\boldsymbol{\psi},\boldsymbol{\upsilon}) + \mathbf{L}\boldsymbol{\phi}, \tag{21a}$$

$$\left(\sqrt{2}\mathbf{E}_{5\times5} + \mathbf{A}^0_{5\times5}\right)\partial_\tau\boldsymbol{\phi} + \mathbf{A}^{\bar{r}}_{5\times5}\partial_{\bar{r}}\boldsymbol{\phi} = \mathbf{B}(\boldsymbol{\upsilon})\boldsymbol{\phi} + \mathbf{M}(\boldsymbol{\psi},\boldsymbol{\varphi}) + \mathbf{N}(\boldsymbol{\varphi},\boldsymbol{\varphi}), \tag{21b}$$

$$\left(\sqrt{2}\mathbf{E}_{3\times3} + \mathbf{A}^0_{3\times3}\right)\partial_\tau\boldsymbol{\varphi} + \mathbf{A}^{\bar{r}}_{3\times3}\partial_{\bar{r}}\boldsymbol{\varphi} = \mathbf{C}(\boldsymbol{\upsilon})\boldsymbol{\varphi}, \tag{21c}$$

$$\left(\sqrt{2}\mathbf{E}_{9\times9} + \mathbf{A}^0_{9\times9}\right)\partial_\tau\boldsymbol{\psi} + \mathbf{A}^{\bar{r}}_{9\times9}\partial_{\bar{r}}\boldsymbol{\psi} = \mathbf{D}(\boldsymbol{\upsilon})\boldsymbol{\psi} + \mathbf{U}(\boldsymbol{\upsilon},\boldsymbol{\varphi}) + \mathbf{V}(\boldsymbol{\upsilon},\boldsymbol{\phi}) + \mathbf{W}(\boldsymbol{\upsilon},\boldsymbol{\upsilon},\boldsymbol{\phi}), \tag{21d}$$

where $\mathbf{K}$ denotes a matrix with constant coefficients, $\mathbf{Q}(\boldsymbol{\upsilon},\boldsymbol{\upsilon})$, $\mathbf{R}(\boldsymbol{\varphi},\boldsymbol{\psi})$ bilinear vector value functions with constant coefficients and $\mathbf{T}(\boldsymbol{\phi},\boldsymbol{\psi},\boldsymbol{\upsilon})$ a trilinear vector valued function with constant coefficients. On the other hand, $\mathbf{L}$ is a linear matrix-valued function with coefficients depending on the coordinates. Furthermore, $\mathbf{E}_{3\times3}$, $\mathbf{E}_{5\times5}$, $\mathbf{E}_{8\times8}$ denote, respectively, the 3×3, 5×5 and 8×8 identity matrices, while $\mathbf{A}^{\bar{s}}_{3\times3}$, $\mathbf{A}^{\bar{s}}_{5\times5}$, $\mathbf{A}^{\bar{s}}_{9\times9}$, $\bar{s}=0,\ldots,3$ are 3×3, 5×5 and 8×8 Hermitian matrices depending on the coordinates. On the other hand $\mathbf{B}(\boldsymbol{\upsilon})$, $\mathbf{C}(\boldsymbol{\upsilon})$, $\mathbf{D}(\boldsymbol{\upsilon})$ denote constant matrix-valued linear function of the entries of $\boldsymbol{\upsilon}$, while $\mathbf{M}(\boldsymbol{\psi},\boldsymbol{\varphi})$, $\mathbf{N}(\boldsymbol{\varphi},\boldsymbol{\varphi})$, $\mathbf{U}(\boldsymbol{\upsilon},\boldsymbol{\varphi})$, $\mathbf{V}(\boldsymbol{\upsilon},\boldsymbol{\phi})$ denote bilinear functions with coordinate dependent coefficients. Finally, $\mathbf{W}(\boldsymbol{\upsilon},\boldsymbol{\upsilon},\boldsymbol{\phi})$ is a trilinear function.

Remark. It can be verified that the evolution system given by equations (21a)-(21d) is *symmetric hyperbolic*. Furthermore, if these evolutions are satisfied, then *the constraint equations implied by the extended conformal Einstein-Maxwell equations propagate* —that is, if they are satisfied on an initial hypersurface, then they are also satisfied at latter time. For details on this see [4, 13].

5 Initial data for static electrovacuum black holes on $\mathcal{C}(\mathbb{S}^3)$

In this section it is discussed how to make use of the time symmetric initial data for electrovacuum black holes discussed in Subsection 2.2.

5.1 Compactification to $\mathbb{S}^3$

The (negative definite) standard metric on $\mathbb{S}^3$, equation (5) is related to the flat metric in spherical coordinates

$$\boldsymbol{\delta} = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2,$$

with (r,θ,φ), $0\leq r<\infty$, $0\leq\theta\leq\pi$ and $0\leq\varphi<2\pi$ via a conformal rescaling and a change of coordinates. More precisely, one has

$$\boldsymbol{\hbar} = -\omega^2\boldsymbol{\delta},$$

with

$$\omega = \frac{2}{\alpha}\sin^2\frac{\psi}{2}, \qquad r(\psi) = \alpha\cot\frac{\psi}{2},$$

and where α is a positive constant which will be fixed in the sequel. This conformal transformation sends the point at infinity in $\mathbb{R}^3$ to the North Pole of $\mathbb{S}^3$ ($\psi=0$), and the origin to the South Pole ($\psi=\pi$). From the discussion in the previous paragraphs and the expression (4a) for the 3-metric $\tilde{\boldsymbol{h}}$ it follows that

$$\tilde{\boldsymbol{h}} = -\phi^2\chi^2\boldsymbol{\delta} = -\phi^2\chi^2\omega^{-2}\boldsymbol{\hbar} = -\Omega^2\boldsymbol{\hbar}, \qquad \Omega\equiv\omega\phi^{-1}\chi^{-1}.$$

A short computation then shows that

$$\Omega = \frac{2}{\alpha}\frac{\sin^2\dfrac{\psi}{2}}{\left(1+\dfrac{(m+q)}{2\alpha}\tan\dfrac{\psi}{2}\right)\left(1+\dfrac{(m-q)}{2\alpha}\tan\dfrac{\psi}{2}\right)}.$$

5.1.1 Non-extremal black holes

In order to simplify the analysis of the Schwarzschild and non-extremal Reissner-Nordström solutions, the constant α will be chosen such that at $\psi=\pi/2$ one has $\mathrm{d}\Omega=0$. This requirement yields

$$\alpha = \tfrac{1}{2}\sqrt{m^2-q^2}, \quad \text{if} \quad m^2>q^2.$$

Thus, one obtains the following conformal factor

$$\Omega = \frac{4}{\sqrt{m^2-q^2}}\frac{\sin^2\dfrac{\psi}{2}}{\left(1+\sqrt{\dfrac{m-q}{m+q}}\tan\dfrac{\psi}{2}\right)\left(1+\sqrt{\dfrac{m+q}{m-q}}\tan\dfrac{\psi}{2}\right)}. \tag{22}$$

In the uncharged case this expression reduces to

$$\Omega = \frac{4}{m} \frac{\sin^2 \dfrac{\psi}{2}}{\left(1 + \tan \dfrac{\psi}{2}\right)^2}. \tag{23}$$

For future reference it is noticed that

$$\Omega = \frac{\psi^2}{\sqrt{m^2 - q^2}} + \mathcal{O}(\psi^3), \qquad \Omega = \frac{(\psi - \pi)^2}{\sqrt{m^2 - q^2}} + \mathcal{O}\left((\psi - \pi)^3\right).$$

Remark. The locus of points for which $\mathrm{d}\Omega = 0$ (i.e. $\psi = \pi/2$) corresponds to the *bifurcation sphere* —a minimal surface for the physical 3-metric $\tilde{\boldsymbol{h}}$.

5.1.2 Extremal black holes

The intrinsic asymmetry of the initial data for the extremal Reissner-Nordström spacetime does not allow to have $\mathrm{d}\Omega = 0$ at $\psi = \pi/2$. Thus, for simplicity, in this case we will set $\alpha = 1$. The conformal factor takes then the form

$$\Omega = \frac{2 \sin^2 \dfrac{\psi}{2}}{\left(1 + m \tan \dfrac{\psi}{2}\right)}. \tag{24}$$

The location of the interior extrema of Ω will depend on the value of Ω via the solution of a cubic equation. The behaviour of the conformal factor at i_1 and i_2 is given, respectively, by

$$\Omega = \tfrac{1}{2}\psi^2 + \mathcal{O}(\psi^3), \qquad \Omega = -\frac{1}{m}(\psi - \pi) + \mathcal{O}\left((\psi - \pi)^2\right). \tag{25}$$

Observation. A short computation with the conformal factor (22) shows that

$$\lim_{q \to \pm m} \Omega = \frac{1}{m} \sin \psi.$$

This conformal factor corresponds to a physical metric $\tilde{\boldsymbol{h}}$ with 2 trumpet asymptotic ends. This 3-metric, together with its initial electric field should give rise to a spacetime with 2 timelike singularities similar to that of the extremal Reissner-Nordström spacetime. This solution does not describe a black hole spacetime as there is no asymptotically Euclidean end and, hence, no null infinity. This computation also shows clarifies why for initial data for the extremal Reissner-Nordström spacetime one cannot use the constant α to have $\mathrm{d}\Omega = 0$ at $\psi = \pi/2$.

5.1.3 The initial electric field

A computation using the formulae of the previous subsection together the transformation rules (18) and the expression (4b) for the initial physical electric field implies that

$$\boldsymbol{E} = -q \csc^2 \psi \, \mathbf{d}\psi, \tag{26}$$

independently of the choice made for the constant α. Notice that this expression is singular at both i_1 and i_2.

5.2 Construction of non-time symmetric data

The triple $(\hbar_{\alpha\beta}, \Omega, E_\alpha)$ with $\hbar_{\alpha\beta}$ given by the line element (5), Ω by either expression (22) or (24) and E_α given by (26) constitute a solution to the conformal constraints, equations (16a)-(16b) and (17), with $\chi_{\alpha\beta} = 0$, $\Sigma = 0$. This initial data naturally leads to a spacetime factor Θ with $\dot{\Theta}_* = 0$ —see expressions (14) and (15) . In order to accommodate conformal factors for which $\dot{\Theta} \neq 0$, one has to consider data sets with non-vanishing Σ. From the transformation rules (18) it follows that if $\Sigma \neq 0$, then the unphysical second fundamental form is non-zero even if $\tilde{\chi}_{\alpha\beta} = 0$. More precisely, one has that

$$\chi_{\alpha\beta} = -\Omega^{-1}\Sigma\, \hbar_{\alpha\beta}. \tag{27}$$

It can be verified that $(\hbar_{\alpha\beta}, \chi_{\alpha\beta}, \Omega, \Sigma, E_\alpha)$ with $\hbar_{\alpha\beta}$, Ω, E_α as given by (5), (22) or (24) and (26), together with a smooth choice of Σ and and associated $\chi_{\alpha\beta}$ given by expression (27) is a solution to the conformal constraints (16a)-(16b) and (17) with $B_\alpha = 0$.

5.3 Initial data for the conformal evolution equations

As discussed in for example [1], given a solution to the conformal Hamiltonian and momentum constraints (16a)-(16b) one can use the conformal constraint equations to construct initial data for the conformal evolution equations —see also [5]. The relevant expressions for the spinorial counterparts of the Weyl and Schouten tensors are given, for the class of initial data sets under consideration, by

$$\begin{aligned}
\hat{P}_{ABCD} &= -\kappa^2\Omega^{-1}D_{(AB}D_{CD)}\Omega - \tfrac{1}{2}\kappa^2 h_{ABCD} - \kappa^2\Omega^2 E_{(AB}E_{CD)} - \kappa^2\Omega^{-1}\Sigma,\\
\phi_{ABCD} &= \kappa^3\Omega^{-2}D_{(AB}D_{CD)}\Omega + \kappa^3\Omega E_{(AB}E_{CD)}.
\end{aligned}$$

Furthermore, one also has that

$$\chi_{ABCD} = \kappa^2\Omega^{-1}\Sigma\, h_{ABCD}, \qquad \varphi_{AB} = \kappa^2 E_{AB},$$

where E_{AB} is the spinorial counterpart of the (unphysical) electric given by expression (26):

$$E_{AB} = -q\csc^2\psi\, x_{AB}.$$

For the remaining conformal fields one has the following initial data:

$$\begin{aligned}
&e^0_{AB} = 0, \qquad e^1_{AB} = \kappa\, x_{AB}, \qquad e^+_{AB} = \kappa\csc\psi\, z_{AB}, \qquad e^-_{AB} = \kappa\csc\psi\, y_{AB},\\
&\xi_{ABCD} = \tfrac{1}{\sqrt{2}}\kappa\cot\psi(\epsilon_{AC}x_{BD} + \epsilon_{BD}x_{AC}) - \tfrac{1}{\sqrt{2}}\left(\epsilon_{AC}\kappa_{BD} + \epsilon_{BD}\kappa_{AC}\right),\\
&f_{AB} = \kappa_{AB},
\end{aligned}$$

with $\kappa_{AB} = \boldsymbol{e}_{AB}\kappa$, and Ω given by expressions (22) or (24). If $\kappa = 1$ and $\Sigma = \mathcal{O}(1)$ at i_1 one has that

$$\phi_{ABCD} = \mathcal{O}(\psi^{-3}), \quad \varphi_{AB} = \mathcal{O}(\psi^{-2}), \quad \chi_{ABCD} = \mathcal{O}(\psi^{-2}).$$

A similar behaviour holds for non-extremal data at i_2 if $\Sigma = \mathcal{O}(1)$ there. For extremal data one has at i_2 that

$$\phi_{ABCD} = \mathcal{O}\left((\psi-\pi)^{-2}\right), \quad \varphi_{AB} = \mathcal{O}\left((\psi-\pi)^{-2}\right), \quad \chi_{ABCD} = \mathcal{O}\left((\psi-\pi)^{-2}\right).$$

This singular behaviour of the initial data at i_1 and i_2 can removed by choosing $\kappa = \mathcal{O}(\psi^2)$ and $\kappa = \mathcal{O}((\psi-\pi)^2)$. *In what follows, for concreteness, the choice*

$$\kappa = \sin\psi$$

will be adopted.

6 The conformal evolution equations for spherically symmetric electrovacuum spacetime

In this section a discussion of the properties of the evolution equations for spherically symmetric electrovacuum spacetimes is provided. In particular, it is shown that the essential dynamics of these evolution equations is governed by a certain subset of equations which will be referred to as the *core system*. The core system allows for simplified numerical and analytical considerations.

6.1 The general spherically symmetric Ansatz

Following up from the discussion of subsection (4.6), one has that the unknowns appearing in the electrovacuum conformal evolution equations (21a)-(21d) are given by the irreducible components of the spinors in (20). These spinors are to be understood as the lift to $\mathcal{C}(\mathbb{S}^3)$ of the corresponding spinors on $\mathbb{S}^3$. As such, they possess a well defined spin-weight. In the present context, spinors having spherical symmetry are invariant under the action of $SU(2,\mathbb{C})$ —in other words, they can only be made up from irreducible components having spin weight zero which vanish upon the application of the operators $\boldsymbol{X}_\pm$. The more general Ansatz for the spinors of (20) satisfying these requirements is given by:

$$e^0_{AB} = e^0 x_{AB}, \quad e^1_{AB} = e^1 x_{AB}, \quad e^+_{AB} = e^+ z_{AB}, \quad e^-_{AB} = e^- y_{AB}, \tag{28a}$$

$$\xi_{ABCD} = \xi_2 \epsilon^2_{ABCD} + \tfrac{1}{3}\xi_h h_{ABCD} + \tfrac{1}{\sqrt{2}}\xi_x(\epsilon_{AC}x_{BD} + \epsilon_{BD}x_{AC}), \tag{28b}$$

$$\chi_{(AB)CD} = \chi_2 \epsilon^2_{ABCD} + \tfrac{1}{3}\chi_h h_{ABCD} + \tfrac{1}{\sqrt{2}}\chi_x(\epsilon_{AC}x_{BD} + \epsilon_{BD}x_{AC}), \tag{28c}$$

$$f_{AB} = f x_{AB}, \tag{28d}$$

$$P_{ABCD} = \theta_2 \epsilon^2_{ABCD} + \tfrac{1}{3}\theta_h h_{ABCD} + \tfrac{1}{\sqrt{2}}\vartheta_x(\epsilon_{AC}x_{BD} + \epsilon_{BD}x_{AC}) + \tfrac{1}{\sqrt{2}}\theta_x \epsilon_{AB}x_{CD}, \tag{28e}$$

$$\phi_{ABCD} = \phi \epsilon^2_{ABCD}, \tag{28f}$$

$$\varphi_{AB} = \varphi x_{AB}, \tag{28g}$$

$$\psi_{ABCD} = \nu_2 \epsilon^2_{ABCD} + \tfrac{1}{3}\nu_h h_{ABCD} + \tfrac{1}{\sqrt{2}}\nu_x(\epsilon_{AC}x_{BD} + \epsilon_{BD}x_{AC}) + \mu_x \epsilon_{AB}x_{CD}, \tag{28h}$$

where the various numerical coefficients in the above expressions have been introduced for convenience. Notice that save for e^+_{AB} and e^-_{AB} which are paired with the operators X_+ and X_-, all the other spinors are constructed by combinations of the spinors x_{AB} and ϵ_{AB}.

In addition, spherical symmetry requires the vanishing of the magnetic parts of ϕ_{ABCD} and φ_{AB}. This is equivalent to requiring ϕ_{ABCD} and φ_{AB} to be real spinors. It follows that

$$\phi = \bar{\phi}, \qquad \varphi = \bar{\varphi}.$$

The substitution of the Ansatz (28a)-(28h) into the conformal evolution equations renders equations for the functions ξ_2, ξ_h, χ_x and ϑ_x which are homogeneous in these unknowns —the precise form of the equations will not be required here. In subsection 6.3.2 it will be seen that the initial data for these evolution equations is given by $\xi_2 = \xi_h = \chi_x = \vartheta_x = 0$ on $\mathcal{C}(\mathbb{S}^3)$. It follows from a standard argument that the homogeneous form of the evolution equations implies

$$\xi_2 = \xi_h = \chi_x = \vartheta_x = 0 \quad \text{for all times.} \tag{29}$$

Thus, they will be dropped from all subsequent considerations.

The unphysical metric

As a consequence of the spherically symmetric Ansatz discussed in the previous subsection, one has that

$$\tau_{PP'}\boldsymbol{\sigma}^{PP'} = \sqrt{2}\left(\mathbf{d}\tau - \frac{e^0_x}{e^1_x}\mathbf{d}\psi\right),$$

$$\boldsymbol{\sigma}^{AB} = -\frac{1}{e^1_x}x^{AB}\mathbf{d}\psi - \frac{2}{e^+_z}y^{AB}\boldsymbol{\alpha}^+ - \frac{2}{e^-_y}z^{AB}\boldsymbol{\alpha}^-.$$

From these expressions, and taking into account expression (19), it follows that the unphysical metric, $\boldsymbol{g}$, is given by

$$\boldsymbol{g} = \mathbf{d}\tau \otimes \mathbf{d}\tau - \frac{e^0_x}{e^1_x}\left(\mathbf{d}\tau \otimes \mathbf{d}\psi + \mathbf{d}\psi \otimes \mathbf{d}\tau\right) - \left(\frac{1}{(e^1_x)^2} - \left(\frac{e^0_x}{e^1_x}\right)^2\right)\mathbf{d}\psi \otimes \mathbf{d}\psi - \frac{1}{e^+_z e^-_y}\mathbf{d}\sigma^2.$$

6.2 The derivatives of the Maxwell spinor

A peculiarity of the electrovacuum evolution system (21a)-(21d) is that it includes the spinorial field ψ_{ABCD} which corresponds to the (Weyl) covariant derivatives of the Maxwell spinor φ_{AB}. In spherical symmetry the non-vanishing components of φ_{AB} can be expressed in terms the Maxwell spinor and the components of the connection. Hence, it is not necessary to consider evolution equations for the spinorial field ψ_{ABCD}. In order to see this one considers the spinorial Maxwell equation rewritten in terms of the Weyl connection, $\hat{\nabla}^Q{}_{A'}\phi_{BQ} = f^Q{}_{A'}\phi_{BQ}$. Combining this equation with the spherically symmetric Ansatz (28a)-(28h) it follows that $\hat{\nabla}^Q{}_{A'}\phi_{BQ} = 0$. Hence, in spherically symmetry one has the symmetry $\psi_A{}^B{}_{CD} = \psi_{(A}{}^B{}_{CD)}$. From here one concludes that

$$\mu_x = -2\sqrt{2}\nu_x, \quad \nu_h = 0.$$

A further computation using the explicit expression of $\nabla_{AA'}\varphi_{BC}$ in terms of the frame derivative $\mathbf{e}_{AA'}\varphi_{CD}$ and connection coefficients together with the Ansatz (28a)-(28h) shows that

$$\nu_2 = -6\left(\xi_x + f\right)\varphi, \qquad \nu_x = \tfrac{1}{6}\left(\chi_2 + 2\chi_h\right)\varphi,$$

where the Maxwell constraint equation $\tau^{PA'}\nabla^Q{}_{A'}\varphi_{PQ} = 0$ has been used to eliminate the "radial" derivative $\partial_\psi\varphi$.

6.3 The spherically symmetric conformal Einstein-Maxwell evolution equations

A lengthy computation using the suite `xAct` for spinorial and tensorial manipulations for `Mathematica` which takes into account the vanishing components given by (29), yields the following *spherically symmetric conformal evolution equations*

$$\partial_\tau e^0 = \tfrac{1}{3}(\chi_2 - \chi_h)e^0 - f, \tag{30a}$$
$$\partial_\tau e^1 = \tfrac{1}{3}(\chi_2 - \chi_h)e^1, \tag{30b}$$
$$\partial_\tau e^\pm = -\tfrac{1}{6}(\chi_2 + 2\chi_h)e^\pm, \tag{30c}$$
$$\partial_\tau f = \tfrac{1}{3}(\chi_2 - \chi_h)f + \theta_x, \tag{30d}$$
$$\partial_\tau \xi_x = -\tfrac{1}{6}(\chi_2 + 2\chi_h)\xi_x - \tfrac{1}{2}\chi_2 f - \theta_x, \tag{30e}$$
$$\partial_\tau \chi_2 = \tfrac{1}{6}(\chi^2 - 4\chi_h)\chi_h - \theta_2 + \Theta\phi, \tag{30f}$$
$$\partial_\tau \chi_h = -\tfrac{1}{6}\chi_2^2 - \tfrac{1}{3}\chi_h^2 - \theta_h - \tfrac{3}{4}\Theta^2\varphi^2, \tag{30g}$$
$$\partial_\tau \theta_x = \tfrac{1}{3}(\chi_2 - \chi_h)\theta_x - \tfrac{1}{3}d_x\phi + \tfrac{1}{2}d_x\Theta\varphi^2 - \tfrac{1}{4}\Theta^2\varphi^2 f, \tag{30h}$$
$$\partial_\tau \theta_2 = \tfrac{1}{6}(\chi_2 - 2\chi_h)\theta_2 - \tfrac{1}{3}\chi_2\theta_h - \dot{\Theta}\phi - \Theta\dot{\Theta}\varphi^2 + \tfrac{3}{4}\Theta^2\chi_2\varphi^2 + \Theta^2\chi_h\varphi^2, \tag{30i}$$
$$\partial_\tau \theta_h = -\tfrac{1}{6}\chi_2\theta_2 - \tfrac{1}{3}\chi_h\theta_h + \tfrac{1}{4}\Theta^2\chi_h\varphi^2 - \Theta\dot{\Theta}\varphi^2, \tag{30j}$$
$$\partial_\tau \phi = -\tfrac{1}{2}(\chi_2 + 2\chi_h)\phi - \tfrac{1}{2}\Theta\chi_2\varphi^2 - \Theta\chi_h\varphi^2 + \dot{\Theta}\varphi^2, \tag{30k}$$
$$\partial_\tau \varphi = -\tfrac{1}{3}\left(\chi_2 + 2\chi_h\right)\varphi. \tag{30l}$$

The most remarkable feature of this system is that it consists entirely of *transport equations* along the congruence of conformal curves upon which our gauge is based. The functions Θ and $\dot{\Theta}$ are explicit functions of the coordinates (τ, ρ) constructed following the discussion of subsection 4.2.2 and formula (15), while $d_x = \kappa^{-1}\Omega'$, where $'$ denotes differentiation with respect to ψ.

6.3.1 The core system of evolution equations

The dynamics of the evolution system (30a)-(30k) is driven by a limited subset of equations. In what follows, let

$$X \equiv \chi_h + \tfrac{1}{2}\chi_2, \qquad P \equiv \theta_h + \tfrac{1}{2}\theta_2.$$

It follows then that equations (30f)-(30g), (30i)-(30j) and (30k) implies the system:

$$\partial_\tau P = -\tfrac{1}{3}XP - \tfrac{1}{2}\dot{\Theta}\phi + \tfrac{3}{4}\Theta^2 X\varphi^2 - \tfrac{3}{2}\Theta\dot{\Theta}\varphi^2, \tag{31a}$$
$$\partial_\tau X = -\tfrac{1}{3}X^2 - P + \tfrac{1}{2}\Theta\phi - \tfrac{3}{4}\Theta^2\varphi^2, \tag{31b}$$
$$\partial_\tau \phi = -X\phi - \Theta X\varphi^2 + \dot{\Theta}\varphi^2, \tag{31c}$$
$$\partial_\tau \varphi = -\tfrac{2}{3}X\varphi. \tag{31d}$$

These equations will be referred to as the *core system*. Notice that these equations are entirely decoupled from the rest of the other evolution equations. The evolution of the remaining field variables can be integrated once the solution to (31a)-(31d) is known. Equation (31a) has the structure of a Riccati equation with source terms. It is the key to understand the dynamics of the core system.

6.3.2 Initial data for the spherically symmetric evolution equations

A direct computation using the expressions (6a) and (7) of Section 3.1.4 shows that

$$D_{AB}\Omega = \Omega' x_{AB},$$
$$D_{(AB}D_{CD)}\Omega = 2(\Omega'' - \cot\psi\Omega')\epsilon^2_{ABCD},$$

where $'$ denotes differentiation with respect to ψ. From these expressions, following the discussion of Section 5.3 and taking into account the choices $\kappa = \sin\psi$, $\Sigma = 0$, it follows that on $\mathcal{C}_\kappa(\mathbb{S}^3)$ one has:

$$e^0_{AB} = 0, \qquad e^1_{AB} = \sin\psi\, x_{AB}, \qquad e^+_{AB} = z_{AB}, \qquad e^-_{AB} = y_{AB}, \tag{32a}$$
$$\xi_{ABCD} = 0, \qquad \chi_{(AB)CD} = 0, \qquad f_{AB} = \cos\psi\, x_{AB}, \tag{32b}$$
$$\Theta_{ABCD} = -2\sin^2\psi\left(\Omega^{-1}\left(\Omega'' - \cot\psi\Omega'\right) + q^2\Omega^2\csc^4\psi\right)\epsilon^2_{ABCD} - \tfrac{1}{2}\sin^2\psi h_{ABCD}, \tag{32c}$$
$$\phi_{ABCD} = 2\sin^3\psi\left(\Omega^2(\Omega'' - \cot\psi\Omega') + q^2\Omega\csc^4\psi\right)\epsilon^2_{ABCD}, \tag{32d}$$
$$\varphi_{AB} = q x_{AB}. \tag{32e}$$

6.4 The*a priori* conformal factor and 1-form $\boldsymbol{d}$

If $\langle \boldsymbol{b}, \dot{x}\rangle_* = 0$, then formula (15) renders an *a priori* conformal factor Θ of the form

$$\Theta = \csc\psi\Omega\left(1 - \frac{\sin^2\psi\Omega'^2}{4\Omega^2}\tau^2\right). \tag{33}$$

The time and space components of the 1-form $\boldsymbol{d}$ are, respectively, given by

$$\dot{\Theta} = -\tfrac{1}{2}\sin\psi\Omega^{-1}\Omega'^2\tau, \qquad d_{AB} = \csc\psi\Omega' x_{AB}.$$

In what follows a discussion of the implications of expression (33) for the various choices of Ω discussed in Sections 5.1.1 and 5.1.2 will be given. In the sequel, all the expressions will be time symmetric. Hence, for simplicity the discussion is it assumed that $\tau \geq 0$. The discussion for $\tau \leq 0$ is analogous.

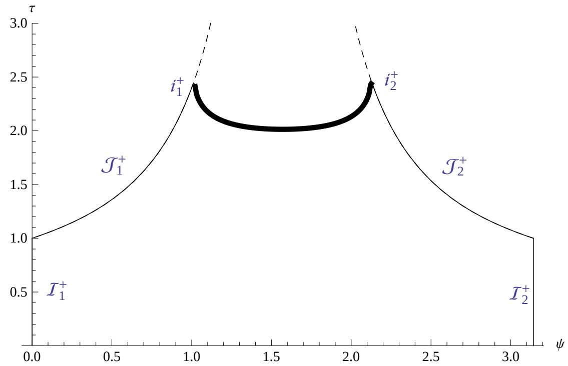

Figure 1: Plot of the *a priori* conformal boundary for the Schwarzschild spacetime with $m = 2$. The dashed lines indicate the parts of the boundary not realised in the spacetime. The thick line corresponds to the location of the singularity, $\mathcal{I}_1^+$ and $\mathcal{I}_2^+$ denote the cylinders at spatial infinity, $\mathscr{I}_1^+$ and $\mathscr{I}_2^+$ correspond to the two components of future null infinity, while i_1^+ and i_2^+ are the future timelike infinities. This plot has been computed numerically.

6.4.1 *A priori* conformal factor for Schwarzschild initial data

For the Schwarzschild spacetime, the conformal factor, Ω, of the initial is given by expressions (23).

The conformal factor satisfies $\Omega = 0$ at $\psi = 0, \pi$. Furthermore, $\mathrm{d}\Omega = 0$ at $\psi = 0, \pi/2, \pi$. By contrast, $\mathrm{d}\Theta_* = \csc\psi \mathrm{d}\Omega \neq 0$ at $\psi = 0, \pi$. The locus of points on $\mathcal{M}_\kappa$ for which $\Theta = 0$ —with Θ as given by equation (33)— can be readily computed —see Figure 1. Notice, however, that not all the points in $\mathscr{I}$ are actually realised —as it will be discussed in the sequel, for certain values of ψ, the solutions to the conformal evolution equations become singular before the conformal boundary is reached. Let $\psi = \psi_{\lightning}$ denote the value of the ψ coordinate of the first conformal geodesic for which the solution to the conformal evolution equations vanishes. Because of the symmetry of Ω, $\psi = \pi - \psi_{\lightning}$ will denote the value of the ψ coordinate of the last conformal geodesic with a singular behaviour of the conformal evolution equations.Then, the set of points for which $\Theta = 0$ with $\psi \in (0, \psi_{\lightning}) \cup (\pi - \psi_{\lightning}, \pi)$ correspond to future null infinity, $\mathscr{I}^+$.

Also of interest are the points for which $\Theta = 0$ and $\mathrm{d}\Theta = 0$ simultaneously. A computation for Θ as given by (33) with Ω given by (23) shows that this occurs if and only if $\psi = 0, \pi$ and $\tau = \pm 1$. These correspond to the *critical sets* where spatial infinity and null infinity meet. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ denote the sets of points for which $|\tau| < 1$ and $\psi = 0, \pi$ —these correspond to the cylinders at spatial infinity first discussed in [5].

6.4.2 *A priori* conformal factor for non-extremal Reissner-Nordström initial data

For the non-extremal Reissner-Nordström spacetime, the conformal factor for the initial data is given by (22).

As in the Schwarzschildean case one has that $\mathrm{d}\Theta \neq 0$ at $\psi = 0, \pi$ unless $\tau = \pm 1$. Again, the locus of points on $\mathcal{M}_\kappa$ for which $\Theta = 0$ can be readily computed —see Figure 2. Again, not all of these points will be realised in the actual solution to the conformal evolution equations.

In the sequel it will be seen that locus of point for which $\Theta = 0$ contains two point located symmetrically about $\pi/2$ at which the solutions to the conformal evolution equation equations are singular. Let $\psi = \psi_{\lightning}, \pi - \psi_{\lightning}$ denote the values of the radial coordinate for which this occurs, and let $\tau_{\lightning}$ be the corresponding value of τ obtained from solving $\Theta = 0$ with $\psi = \psi_{\lightning}, \pi - \psi_{\lightning}$. As in the Schwarzschildean case, the set of points for which $\Theta = 0$ and $\psi \in (0, \psi_{\lightning}) \cup (\pi - \psi_{\lightning}, \pi)$ correspond to future null infinity, $\mathscr{I}^+$. The set of points, $\mathscr{H}^+$, for which $\Theta = 0$ with $\psi \in (\psi_{\lightning}, \pi - \psi_{\lightning})$ also belong to the *a priori* conformal boundary. However, these cannot be considered as belonging to future null infinity —instead, they correspond to the *Cauchy horizon.*

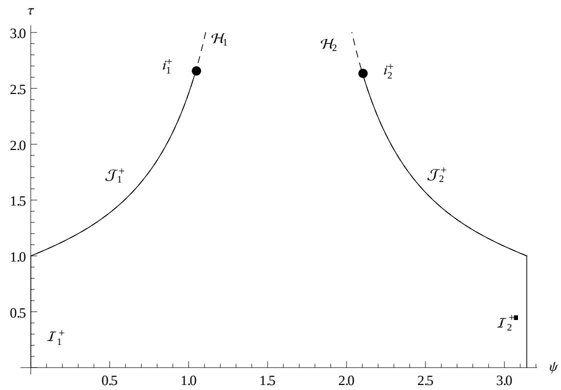

Figure 2: Plot of the *a priori* conformal boundary for the non-extremal Reissner-Nordström spacetime with $m = 2$, $q = -1$. The dashed lines indicate the parts of the *a priori* boundary corresponding to the Cauchy horizon. The cylinders at spatial infinity are denoted by $\mathcal{I}_1^+$ and $\mathcal{I}_2^+$, $\mathscr{I}_1^+$ and $\mathscr{I}_2^+$ correspond to the two components of future null infinity, while i_1^+ and i_2^+ are the future timelike infinities —the starting point of the timelike singularities. This plot has been computed numerically.

As in the case of the Schwarzschild spacetime, the critical sets where spatial infinity and null infinity meet are given by the conditions $\Theta = 0$ and $\mathrm{d}\Theta = 0$ simultaneously. A computation shows, again, that this occurs if and only if $\psi = 0$, π and $\tau = \pm 1$. Anew, we define the cylinders at spatial infinity $\mathcal{I}_1$ and $\mathcal{I}_2$ by the conditions $|\tau| < 1$, $\psi = 0$, π.

6.4.3 *A priori* conformal factor for extremal Reissner-Nordström initial data

In the case of the extremal Reissner-Nordström spacetime, the conformal factor for the initial data is given by equation (24). A plot of the corresponding *a priori* conformal boundary is given in Figure 3.

Close to $\psi = 0$, the conformal factor Ω behaves like the one of the Schwarzschild and non-extremal Reissner-Nordstróm spacetimes. In particular at $\psi = 0$, $\tau = \pm 1$ one has as in those cases, the critical set given by $\Theta = 0$, $\mathbf{d}\Theta = 0$, where spatial infinity meets null infinity. The cylinder at spatial infinity $\mathcal{I}_1$ (where $\Theta = 0$) is given by the conditions $|\tau| < 1$, $\psi = 2$. The situation at the trumpet end $\psi = \pi$ is, by contrast, completely different. From the expansions (25) it follows that although $\Omega = 0$ at $\psi = \pi$, one nevertheless has that $\Theta_* = \csc\psi\Omega \neq 0$ at $\psi = \pi$. It follows then from formula (33) that $\Theta = 0$ at $\psi = \pi$ only if $\tau = \pm 2$. A direct computation shows that $\mathbf{d}\Theta = 0$ at $\tau = \pm 2$, $\psi = \pi$. However, in this case one does not have a critical set nor a cylinder at spatial infinity as $\Theta \neq 0$ for $\psi = \pi$, $\tau \in (-2, 2)$.

Let ψ_{i^+} denote the value of the ψ coordinate for which $\mathbf{d}\,\Omega = 0$ —recall that in the extremal case this point depends on m. The set of points for which $\Theta = 0$ with $\psi \in (0, \psi_{i^+})$, $\tau > 0$ will correspond to future null infinity, $\mathscr{I}^+$. The set of points, $\mathscr{H}^+$, for which $\Theta = 0$ with $\tau > 0$, $\psi \in (\psi_{i^+}, \pi]$ although by definition belonging to the conformal boundary, cannot be regarded as belonging to future null infinity, but have to be identified with the Cauchy horizon. The value $\psi = \psi_{i^+}$ corresponds to the spatial location of timelike infinity, i^+, which because of the gauge choice $\langle \boldsymbol{b}, \dot{x} \rangle_* = 0$ has no finite coordinate location in this conformal representation.

Notice, finally, that in this conformal representation the point at infinity in the trumpet asymptotic end i_2, with coordinates $(\tau = 0, \psi = \pi)$ is disconnected from the starting point of the Cauchy horizon $(\tau = 2, \psi = \pi)$. The standard Penrose diagram of the extremal Reissner-Nordström spacetime indicates that the timelike singularity of the spacetime starts at the point at infinity in the trumpet end. However, as it will be seen in the sequel, in the present representation the numerical integration of the conformal evolution equations along the conformal geodesic with $\psi = \pi$ does not show the existence of singularities.

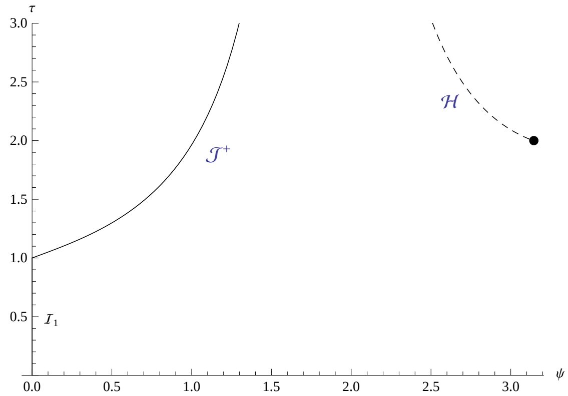

Figure 3: Plot of the *a priori* conformal boundary for the extremal Reissner-Nordström spacetime with $m = 2$, $q = -2$. The dashed curve indicates the part of this boundary that is to be identified with the Cauchy Horizon, $\mathcal{H}$ —notice that this curve is disconnected from the initial hypersurface. The cylinder of the asymptotically Euclidean end is denote by $\mathscr{I}$. This conformal representation does not have a finite location for future timelike infinity, i^+.

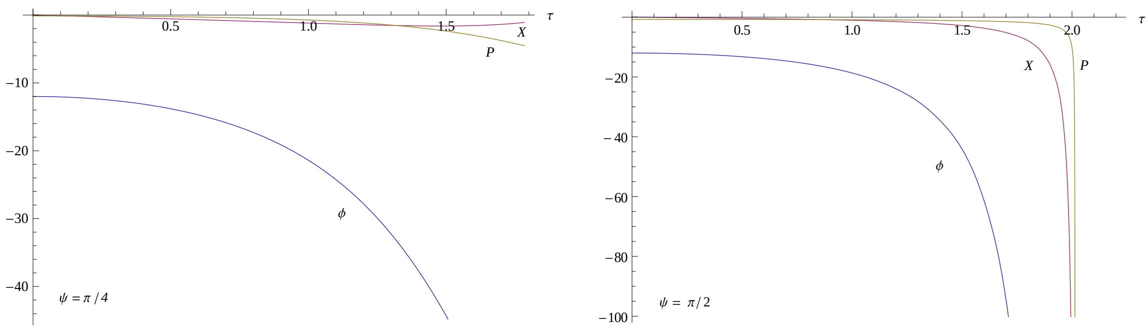

Figure 4: Plots for the numerically constructed solutions to the core system for the Schwarzschild spacetime with $m = 2$ along the conformal geodesics described by the conditions $\psi = \pi/4$ and $\psi = \pi/2$. The conformal geodesic with $\psi = \pi/4$ reaches null infinity at $\tau \approx 1.7836$. The conformal geodesic with $\psi = \pi/2$ blows up at $\tau \approx 2.0137$. The value of $\psi_\lightning$ is found to lie in the interval $(1.0086, 1.0192)$.

7 Numerical solutions to the evolution equations

In this section we discuss numerically constructed solutions to the core evolution equations (31a)-(31d) for initial data sets with choices of the parameters m and q corresponding, respectively, to data for the Schwarzschild, non-extremal and extremal Reissner-Nordström spacetimes. For simplicity, generalised conformal Gaussian systems with $\langle \boldsymbol{b}, \dot{x} \rangle_* = 0$. The numerical computations have been carried out with the numerical integrator routine for ordinary difference equations of Mathematica which is accurate enough for the purposes of the present analysis.

7.1 The Schwarzschild spacetime

Consistent with the discussion of Section 6.4.1, the numerical integration of the core system equations (31a)-(31c) for initial data for the Schwarzschild spacetime reveals the existence of two types of conformal geodesics. The first class consists of conformal geodesics for which $\psi \in [0, \psi_\lightning) \cup (\pi - \psi_\lightning, \pi]$, the solutions of the core system remain regular and bounded up to the value of τ for which $\Theta = 0$. Furthermore, integration can be continued past this value of τ to yield a solution which remains regular. The second class of conformal geodesics is given by the condition $\psi \in [\psi_\lightning, \pi - \psi_\lightning]$. In this case the numerically constructed solutions blow up before one reaches the value of τ for which $\Theta = 0$. These computations are in agreement with those reported in

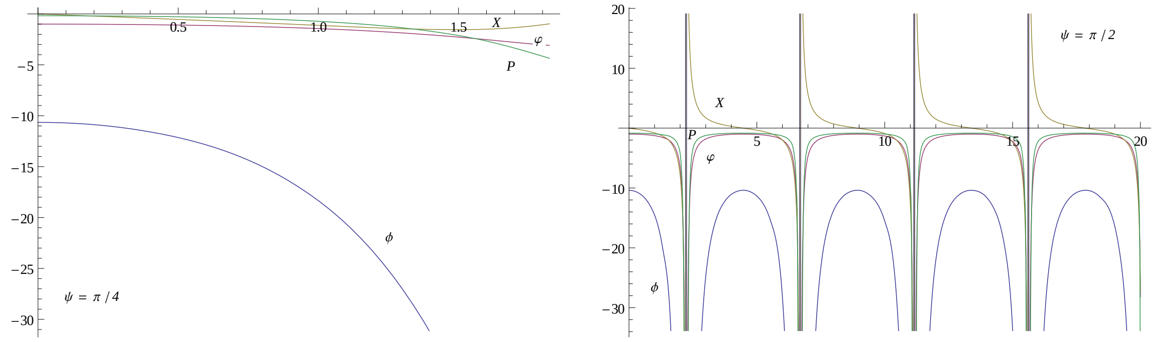

Figure 5: Plots of the numerically constructed solutions to the core system for the non-extremal Reissner-Nordstróm spacetime with $m = 2$ and $q = -1$ along the conformal curves described by the conditions $\psi = \pi/4$ and $\psi = \pi/2$. The conformal curve with $\psi = \pi/4$ reaches null infinity at approximately $\tau \approx 1.8242$. The conformal geodesic with $\psi = \pi/2$ can be integrated up to arbitrarily large values of τ and exhibits the strong oscillatory behaviour first discussed in [8]. This solution is at all times regular —the graph has been cut in the τ-axis for the ease of presentation. The value of $\psi_{\natural}$ is found to lie in the interval $(1.040, 1.0507)$.

[17, 18].

Figure 4 provides plots of the numerically constructed solutions of the core system for two prototypical conformal geodesics for the Schwarzschild spacetime with $m = 2$. Similar behaviour is observed for other values of m.

7.2 The non-extremal Reissner-Nordström spacetime

Consistent with the discussion in Section 6.4.2, the numerical integration of the solutions to the core system (31a)-(31d) reveals two different behaviours of the solution according to whether $\psi \in [0, \psi_{\natural}) \cup (\pi - \psi_{\natural}, \pi]$ or $\psi \in (\psi_{\natural}, \pi - \psi_{\natural})$. In the case of conformal curves with $\psi \in [0, \psi_{\natural}) \cup (\pi - \psi_{\natural}, \pi]$ one obtains a behaviour similar to that of analogous curves in the Schwarzschild spacetime: the numerically constructed solutions to the core system remain regular up to and beyond the corresponding value of τ corresponding to the location of future null infinity. For conformal curves with $\psi \in (\psi_{\natural}, \pi - \psi_{\natural})$, the numerically constructed solutions are found to be regular, but present a very strong oscillatory behaviour. The number of oscillations increases as one approaches to the conformal curve with $\psi = \pi/2$ —where, presumably, one has an infinite number of these. This oscillatory behaviour was first discussed in [8]. Presumably, numerically solutions along the conformal curves with $\psi_{\natural}$ and $\pi - \psi_{\natural}$ blow up in finite time in the same way as, say, the solutions along the conformal geodesic with $\psi = \pi/2$ in the Schwarzschild spacetime.

Figure 5 provides plots of the numerically constructed solutions to the core system for two prototypical conformal curves in the non-extremal Reissner-Nordström spacetime with $m = 2$, $q = -1$. Similar behaviour is observed for other values of the parameters.

7.3 The extremal Reissner-Nordström spacetime

Consistent with the discussion in Section 6.4.3, the behaviour of the solutions to the core system for extremal Reissner-Nordström spacetime can be grouped in two categories. The first corresponds to those conformal curves with $\psi \in [0, \psi_{i^+})$, for where the solutions remain regular up to and beyond future null infinity. The other second one is that of conformal curves with $\psi \in (\psi_{i^+}, \pi]$. The solutions along these conformal geodesics remain, again, regular along the conformal curves, up to and beyond the Cauchy horizon. *This is the case even for the conformal curve with $\psi = \pi$, corresponding to the location of the timelike singularity in the physical extremal Reissner-Nordström spacetime.* This is a remarkable property of this conformal representation of the spacetime and may prove of great utility in applications. The two regimes are separated by

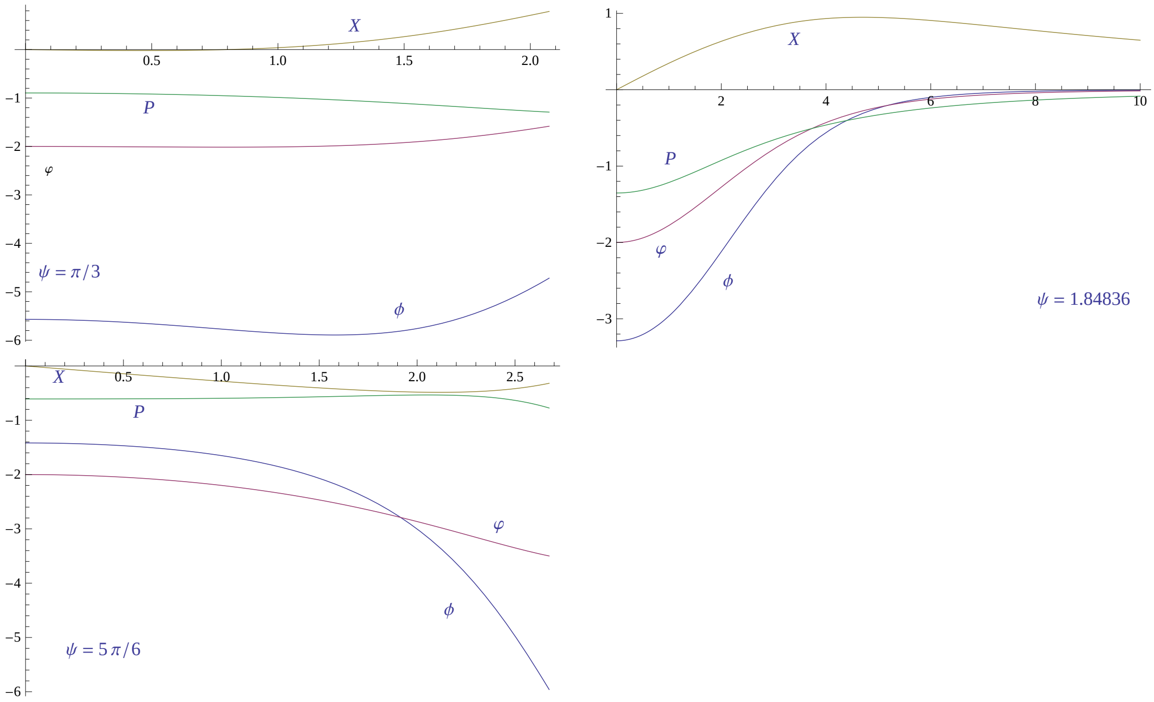

Figure 6: Plots of the numerically constructed solutions to the core system for the extrema Reissner-Nordström spacetime with $m = 2$, $q = -2$ along the conformal curves with $\psi = \pi/3, 1.8484, 5\pi/6$. For the conformal curve with $\psi = \pi/3$, the integration was carried out up to future null infinity at $\tau \approx 2.0745$. The conformal curve with $\psi = 1.8484$ is the one passing through future timelike infinity —the solutions along this curve can be integrated up to arbitrarily large value of τ. The solutions to the core system along the conformal curve with $\psi = 5\pi/6$ are integrated up to the location of the Cauchy horizon at $\tau \approx 2.6742$.

the conformal curve *passing through future timelike infinity* given by $\psi = \psi_{i^+}$. Along this curve one observes a very fast decay of the Weyl and Maxwell spinors.

Figure 6 provides plots of the numerically constructed solutions of the core system along three different conformal curves of the extremal Reissner-Nordström with $m = 2$, $q = -2$. Similar behaviour is observed for other values of the parameters.

8 Behaviour on selected conformal curves

The structure of the core evolution system suggests the possibility of carrying out an analytic discussion of the various fields along certain selected conformal curves: those starting at the spatial infinities and at interior points where $\mathrm{d}\Theta_* = 0$. The analysis presented is of potential relevance in applications and serve, also, as check of the numeric computations.

8.1 The Schwarzschild and non-extreme Reissner-Nordström spacetimes

8.1.1 Conformal curves starting at the spatial infinities

At the spatial infinities of the Schwarzschild and non-extremal Reissner-Nordström one has that $\Theta = \dot{\Theta} = 0$. Hence, the core evolution equations reduce to

$$\partial_\tau P = -\tfrac{1}{3} X P, \tag{34a}$$
$$\partial_\tau X = -\tfrac{1}{3} X^2 - P, \tag{34b}$$
$$\partial_\tau \phi = -X\phi, \tag{34c}$$
$$\partial_\tau \varphi = -\tfrac{2}{3} X\varphi. \tag{34d}$$

The initial data for these equations is given by

$$P(0) = -\tfrac{1}{2}, \quad X(0) = 0, \quad \phi(0) = -6m, \quad \varphi(0) = q.$$

It can be readily verified that the solution to equations (34a)-(34d) is given by

$$P(\tau) = -\tfrac{1}{2}, \quad X(\tau) = 0, \quad \phi(\tau) = -6m, \quad \varphi(\tau) = q.$$

That is, one has a (constant) solution which valid for all times. This solution also holds for the asymptotic flat end of the extremal Reissner-Nordström spacetime.

8.1.2 Conformal curves starting at the bifurcation sphere of the Schwarzschild spacetime

As discussed in Section 5.1.1, the location of the bifurcation sphere is characterised by the conditions $\mathrm{d}\Omega = 0$, $\Omega \neq 0$. For simplicity, it is assumed that $\langle b, \dot{x}\rangle_* = 0$. Thus, it follows that $\dot{\Theta} = 0$ along the conformal geodesic passing through it. The corresponding core evolution equations are given by

$$\partial_\tau P = -\tfrac{1}{3} X P, \tag{35a}$$
$$\partial_\tau X = -\tfrac{1}{3} X^2 - P + \tfrac{1}{2}\Theta\phi, \tag{35b}$$
$$\partial_\tau \phi = -X\phi. \tag{35c}$$

The conformal factor Θ is constant along the conformal geodesic under consideration. The corresponding initial data is given, in this case, by

$$P(0) = -\tfrac{3}{4}, \quad X(0) = 0, \quad \phi(0) = -6m. \tag{36}$$

In the sequel, it is shown that this solution must blow-up in finite time.

Standard theorems on ordinary differential equations ensure the existence of a solution to (35a)-(35c) with the data (36) for at least a $\tau \in [0, T)$ with T sufficiently small. The solutions to equations (35a) and (35c) can be written as

$$P(\tau) = P_0 \zeta^{1/3}(\tau), \quad \phi(\tau) \equiv \phi_0 \zeta(\tau),$$

with

$$\zeta(\tau) \equiv e^{-\int_0^\tau X(s) ds}. \tag{37}$$

As a consequence of these expressions and the initial data (36), one has that as long as a solution exists, $P(\tau)$ and $\phi(\tau)$ must be negative. Also, as a consequence of the initial data one has that $\dot{X}(0) < 0$. Thus, if T is sufficiently small, one has that $X < 0$ in $[0, T)$. Notice that $\dot{P} < 0$ and $\dot{\phi} < 0$ on $(0, T)$, so that $P_0 > P$ and $\phi_0 > \phi$. It follows that

$$\dot{X} = -\tfrac{1}{3} X^2 - P + \tfrac{1}{2}\Theta\phi \leq -\tfrac{1}{3} X^2 - P_0 + \tfrac{1}{2}\Theta\phi_0, \quad \text{on } [0, T). \tag{38}$$

Letting

$$c^2 \equiv P_0 - \tfrac{1}{2}\Theta\phi_0 > 0,$$

it follows from integration of the inequality (38) that

$$X \leq -\sqrt{3}\alpha \tan\left(\frac{\alpha}{\sqrt{3}}\tau\right) < 0.$$

Now,

$$\tan\left(\frac{\alpha}{\sqrt{3}}\tau\right) \to -\infty, \qquad \text{as } \tau \to \frac{\sqrt{3}}{2\alpha}\pi.$$

Hence, X must also blow up in finite time.

9 Conclusions

The numerical simulations discussed in the present article provide evidence for the following conjecture:

Conjecture. *The conformal representation of the extremal Reissner-Nordström spacetime given by the solutions of the evolution equations* (30a)-(30l) *is regular for* $\tau \in [0, \infty)$. *Furthermore, there also exists a similarly regular conformal representation in which* i^+ *has a finite location.*

A proof of this conjecture requires a detailed analysis of the solutions to the evolution system (30a)-(30l) for the various values of ψ. Such a representation may be of potential relevance in a proof of the stability of the extremal Reissner-Nordström spacetime by conformal methods.

Acknowledgements

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