

1. [First-order logic] Let (G, \cdot) be a group. An element $g \in G$ is called *torsion* if g is of finite order, i.e., there exist an n such that $g^n = 1$. A group G is *torsion* if every element is torsion. A group G is *torsion-free* if every element different from 1 is of infinite order.

An abelian group $(A, +)$ is *divisible* if for every element $a \in A$ and every $n > 0$ there exists an $x \in A$ such that $nx = \underbrace{x + \cdots + x}_{n \text{ times}} = a$.

- (a) Find a first-order axiomatisation for the class of torsion-free divisible abelian groups and prove that this theory is complete.
- (b) Prove that the property ‘ G is torsion’ cannot be axiomatised in a first-order way in the language of groups.

2. [Algorithm for QE in ACF] In the lecture we gave an indirect proof of quantifier elimination for algebraically closed fields. However, there exist various algorithms for explicit elimination of quantifiers. By studying an elementary exposition of one of them:

http://cs.nyu.edu/courses/spring04/G22.3033-003/lec/lec10_h4.pdf

follow through the procedure of eliminating the quantifiers from:

$$\exists w \exists x \exists y \exists z (aw + by = 1 \wedge ax + bz = 0 \wedge cw + dy = 0 \wedge cx + dz = 1)$$

3. [Lefschetz principle] Let $\sigma : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be an algebraic automorphism of \mathbb{C}^n viewed as an algebraic variety (the affine n -space, sometimes denoted \mathbb{A}^n). In other words, the components of σ are polynomial maps. Prove the following statement.

If $\sigma^2 = 1$, then σ has a fixed point.

4. [Morley rank in topological terms] Let M be an \aleph_0 -saturated model of a complete theory T . There is a notion of *Cantor-Bendixson derivative* of a topological space, see:

[http://en.wikipedia.org/wiki/Derived_set_\(mathematics\)](http://en.wikipedia.org/wiki/Derived_set_(mathematics))

We shall apply these considerations to the Stone space $X = S_n(M)$. Given a type $p \in X$, we say that its *Cantor-Bendixson rank* is α , written $CB(p) = \alpha$, if $p \in X^\alpha \setminus X^{\alpha+1}$.

Prove the following:

- (a) $MR(p) \geq CB(p)$;
- (b) if T is totally transcendental, $MR(p) = CB(p)$.

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¹ Please send me the solutions by email—scanned manuscripts are acceptable!