# CW5 SOLUTIONS

① (a) and (c) satisfy all requirements to be functions.

(b) is NOT, since it attempts to assign two values to 1, and it doesn't determine the value for 5.

(d) is NOT, it doesn't determine the value for 5.

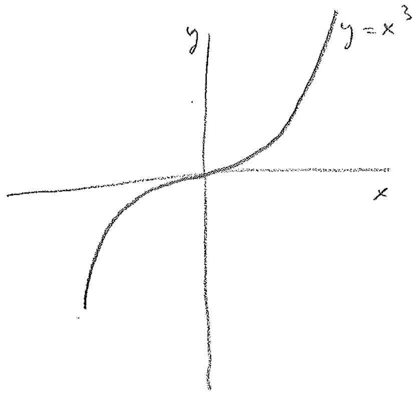
② (a) NOT 1-1 since $f(1) = f(2)$

NOT ONTO since the value 1 is never attained.

(b) $g$ is 1-1 because no two arguments are assigned the same value.

$g$ is ONTO because all values from $\{1,2,3,4,5\}$ are attained.

(c) Graph of $h$:

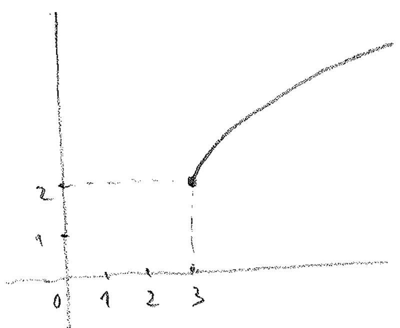



$h$ is 1-1 because each horizontal line intersects the graph of $h$ in at most one point, i.e.

$h(x_1) = h(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

$h$ is ONTO because each horizontal line intersects the graph of $h$ in at least one point, i.e. for every $y$, there exists an $x$ s.t. $h(x) = y$; indeed, given $y \in \mathbb{R}$, take $x = \sqrt[3]{y}$; then $h(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y$.

(d) Graph of $k$:




k is 1-1: Suppose $k(x_1) = k(x_2)$

$\sqrt{x_1 + 1} = \sqrt{x_2 + 1} \quad / ()^2$

$x_1 + 1 = x_2 + 1 \quad / -1$

$x_1 = x_2$

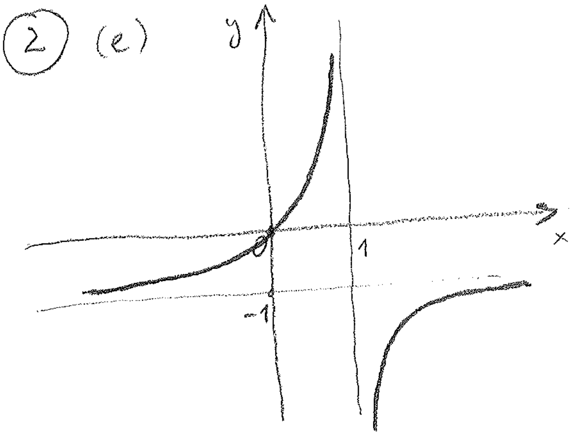
k is ONTO: Given $y \geq 2$, need to find $x \geq 3$ s.t. $k(x) = y$, i.e. $\sqrt{x+1} = y$

$\Rightarrow x + 1 = y^2$

$x = y^2 - 1$

Clearly, since $y \geq 2$, $x = y^2 - 1 \geq 3$. ✓

(2) (e)




$j$ is 1-1 : Suppose $j(x_1) = j(x_2)$

$$\Rightarrow \quad \frac{1+x_1}{1-x_1} = \frac{1+x_2}{1-x_2} \quad / \cdot (1-x_1)(1-x_2)$$

$$(1+x_1)(1-x_2) = (1-x_1)(1+x_2)$$

$$\cancel{1} + x_1 - x_2 - \cancel{x_1 x_2} = \cancel{1} - x_1 + x_2 - \cancel{x_1 x_2}$$

$$\Rightarrow \quad 2x_1 = 2x_2$$

$$\Rightarrow \quad x_1 = x_2$$

$j$ is ONTO : Given $y \neq -1$, need to find $x \neq 1$ s.t. $j(x) = y$, i.e.

$$\frac{1+x}{1-x} = y \quad / \cdot (1-x) \quad \Rightarrow \quad 1 + x = y(1-x)$$

$$1 + x = y - xy, \qquad x(1+y) = y - 1$$

$$\Rightarrow \quad x = \frac{y-1}{y+1}.$$

CHECK : $$j\left(\frac{y-1}{y+1}\right) = \frac{1 + \frac{y-1}{y+1}}{1 - \frac{y-1}{y+1}} = \frac{\frac{y + \cancel{1} + y - \cancel{1}}{\cancel{y+1}}}{\frac{\cancel{y} + 1 - \cancel{y} + 1}{\cancel{y+1}}} = \frac{2y}{2} = y \quad ✓$$

(3) Invertible functions are those which are both 1-1 and ONTO, i.e. BIJECTIVE. Those appear in (b), (c), (d), (e), and we have already found:

$$h^{-1}(y) = \sqrt[3]{y}$$

$$k^{-1}(y) = y^2 - 1$$

$$j^{-1}(y) = \frac{y-1}{y+1}$$

$g^{-1}$:

| Y | X |
|---|---|
| 5 | 1 |
| 4 | 2 |
| 3 | 3 |
| 2 | 4 |
| 1 | 5 |

NOTE: $g^{-1} = g$.

(4)

$f$:

| X | Y |
|---|---|
| 1 | 2 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |
| 5 | 5 |

$g$:

| X | Y |
|---|---|
| 1 | 5 |
| 2 | 4 |
| 3 | 3 |
| 4 | 2 |
| 5 | 1 |

(a) $f \circ g$:

| | |
|---|---|
| 1 | 5 |
| 2 | 4 |
| 3 | 3 |
| 4 | 2 |
| 5 | 2 |

(b) $g \circ f$:

| | |
|---|---|
| 1 | 4 |
| 2 | 4 |
| 3 | 3 |
| 4 | 2 |
| 5 | 1 |

(c) $f^{-1} \circ g^{-1}$ doesn't exist because $f$ is not invertible.

(4) (d) Since the codomain of $j$ is $\mathbb{R}\setminus\{-1\} \subseteq \mathbb{R}$ - the domain of $h$, the composite $h\circ j$ exists as a function $\mathbb{R}\setminus\{1\} \longrightarrow \mathbb{R}$

$$(h\circ j)(x) = h(j(x)) = (j(x))^3 = \left(\frac{1+x}{1-x}\right)^3$$

(e) Since the codomain $\mathbb{R}$ of $h$ is NOT CONTAINED in the domain $\mathbb{R}\setminus\{1\}$ of $j$, the composite $j\circ h$ DOES NOT MAKE SENSE.

In fact, if we formally compute,

$$(j\circ h)(x) = j(h(x)) = \frac{1+h(x)}{1-h(x)} = \frac{1+x^3}{1-x^3},$$ this is not defined for $x=1$.

(5) (a) Codomain of $g$ is $\mathbb{R}$, domain of $f$ is $\mathbb{R}$, so $f\circ g$ makes sense as a function $\{x\in\mathbb{R} : x\geq 1\} \longrightarrow \{x\in\mathbb{R} : x\geq 1\}$:

$$(f\circ g)(x) = f(g(x)) = \sqrt{g(x)^2+1} = \sqrt{\left(\sqrt{x^2-1}\right)^2+1} = \sqrt{x^2-1+1} = \sqrt{x^2} = |x| = x$$

as $x\geq 1$.

(b) Codomain of $f$ is $\{x\in\mathbb{R} : x\geq 1\}$, domain of $g$ is $\{x\in\mathbb{R} : x\geq 1\}$, so $g\circ f$ makes sense as a function $\mathbb{R}\longrightarrow\mathbb{R}$:

$$(g\circ f)(x) = g(f(x)) = \sqrt{f(x)^2-1} = \sqrt{\left(\sqrt{x^2+1}\right)^2-1} = \sqrt{x^2+1-1} = \sqrt{x^2} = |x|$$

Where the absolute value is

$$|x| = \begin{cases} x, & \text{when } x\geq 0 \\ -x, & \text{when } x<0 \end{cases}$$