Topology

Exercise sheet 9

There are two sections. It would be good if you looked at the questions in Section 1 before the next lecture. We shall go through these on Tuesday in the tutorial.

1. Suppose that $r\colon [0,1] \to [0,1]$ is any continuous function with $r(0) = 0$ and $r(1) = 1$, and that α is any path. Prove that the path β given by $\beta(t) = \alpha(r(t))$ is homotopic to α.

2. Suppose that $f\colon X \to Y$ is a continuous map and $x \in X$. We can define a map from $P_1(X, x) \to P_1(Y, f(x))$ by sending a loop α in X to the loop $f \circ \alpha$ in Y. Prove that this induces a map f_* from $\pi_1(X, x)$ to $\pi_1(Y, f(x))$ and that this map is a group homomorphism.

   Now suppose that f is homeomorphism. Prove that f_* is an isomorphism.

3. [In this question we generalise the idea of homotopies. It is a little harder/longer so don't worry if you can't do it. Also the definitions in this question are NOT examinable but may be used in other questions on exercise sheets.]

   Suppose that $f, g\colon X \to Y$ are continuous functions. We say f, g are homotopic if there is a continuous function $F\colon X \times [0,1] \to Y$ with $F(s, 0) = f(s)$ and $F(s, 1) = g(s)$. [This more general definition is why we need to say "relative $\{0, 1\}$" all the time when working with paths; in general we do not require any points to be fixed.]

   Define C to be the constant map mapping of all of X to x, and let I denote the identity map (i.e., $I(x) = x$ for all x). Suppose that C and I are homotopic. Prove that X is simply connected.

Second Section

4. Extend Question 4. Suppose that X and Y are two topological spaces and $f\colon X \to Y$ and $g\colon Y \to X$ have the property that $g \circ f$ is homotopic to the identity map I_X on X and $f \circ g$ is homotopic to the identity map I_Y on Y. In this case we say X and Y are homotopy equivalent spaces.

   (a) Check that homeomorphic spaces are homotopy equivalent.

   (b) Show that $\mathbb{R}^n$ is homotopy equivalent to the single point space (i.e., to the unique topological space with a single point).

   (c) Suppose X and Y are homotopy equivalent and X is simply connected. Prove that Y is simply connected.

   (d) Convince yourself that 'homotopy equivalent' is an equivalence relation on topological spaces.