Topology
Exercise sheet 8

It would be good if you looked at these questions before the the tutorial on Tuesday. Pay a particular attention to properties 1, 2, 3 listed below since they play a crucial role in the construction of the so called fundamental group which will be the main topic of the week 10 lectures. Hint: the proofs in question are often based on the understanding of the geometric meaning of the definition of the equivalence relation $\simeq$. It is therefore useful to draw a picture which supports the proof.

1. Prove that $c_x \cdot \alpha \simeq \alpha$.
2. Prove that $(\alpha \cdot \beta) \cdot \gamma \simeq (\alpha \cdot \beta) \cdot \gamma$.
3. Suppose X is a path connected topological space and that $x, y \in X$. Prove that every loop based at x is homotopic to c_x if and only if every loop based at y is homotopic to c_y. Hint: suppose γ is a path from x to y and consider paths of the form $\gamma^{-1} \cdot \alpha \cdot \gamma$ (and maybe more complicated expressions too).
4. Prove that $A = \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\}$ is simply connected. (One way would be to show that A is homeomorphic to something simply connected.)
5. Give an example of a simply connected space X and a continuous surjection $f \colon X \to Y$ with Y not simply connected. (No proof required.) Are all paths in Y the image of paths in X?
6. In this question we aim to prove that S^1 is simply connected if and only if $R^2 \setminus \{0\}$ is simply connected. Later in the course we shall show that neither is simply connected but in this question we only prove that they are the same.
   (a) Suppose, that S^1 is simply connected and that α is a loop in $\mathbb{R}^2 \setminus \{0\}$. Find a loop β in S^1 with $\beta \simeq \alpha$. Deduce that $\mathbb{R}^2 \setminus \{0\}$ is simply connected.
   (b) Now conversely suppose that $\mathbb{R}^2 \setminus \{0\}$ is simply connected and that α is a loop in S^1. We know that there is a homotopy F between α and c_x for some x.
      i. Why does this not immediately show S^1 is simply connected?
      ii. Modify the homotopy to show that S^1 is simply connected.