Topology
Exercise sheet 6

There are many questions: you do not need to do them all but you should think whether you could do them. Ideally, if I asked you in the tutorial how to do a question you would be able to answer it. They are in no particular order so if you can't do one go on to the next.

1. [This question is not on recent material but it will be very helpful for the next lecture.]

   Suppose that α, β and γ are all paths in X and $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. Are the two paths $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ the same? (In other words I am asking is the join operation on paths associative.)

2. [Less important.]

   Suppose that (X, d) is a compact metric space and that $f \colon X \to \mathbb{R}$ is continuous. By finding a suitable open cover and applying Lemma 23 prove that f is uniformly continuous.

   Remember that a function f is uniformly continuous if

   $$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in X, d(x,y) < \delta\ :\ |f(x) - f(y)| < \varepsilon.$$

3. In this question we check some easy properties of the product topology.

   (a) Suppose X is a topological space and Y is the topological space on the single point y (there is only one such space). Check that $X \times Y$ and X are homeomorphic.

   (b) Suppose X and Y are topological spaces and $A \subset X$. We can view $A \times Y$ as a topological space in two ways: as the product of the topological space A (with the subspace topology from X) and Y, or as the subspace $A \times Y$ of the product space $X \times Y$. Check these give the same topology.

   (c) Suppose that X and Y are non-empty topological spaces. Prove that $X \times Y$ is Hausdorff if and only if X and Y are both Hausdorff.

4. Prove that the metric topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the same as the product topology *[hint: use Lemma 24 (the one giving the alternative description of the product topology).]*

   What about $\mathbb{R}^2$ with the 'Manhattan' or l_1 metric:

   $$d((x,y),(x',y')) = |x - x'| + |y - y'|$$

5. Suppose (X, d_X) and (Y, d_y) are metric spaces. Prove that the metric on $X \times Y$ given by

   $$d((x,y),(x',y')) = \max(d_X(x,x'), d_Y(y,y'))$$

   gives a metric which gives the product topology on $X \times Y$.

   [Remark: we could also take $d((x,y),(x',y')) = d_X(x,x') + d_Y(y,y')$ or $d((x,y),(x',y')) = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2}$.]

6. Suppose that $(X, \mathcal{F})$ is a topological space and that $\sim$ is an equivalence relation on X. In each of the following cases work out the possibilities for the quotient toplogy on $X/\sim$.

   (a) $\mathcal{F}$ is the discrete topology on X.

   (b) $\mathcal{F}$ is the indiscrete topology on X.

   (c) $\mathcal{F}$ is the co-finite topology on X.

7. Consider the topological space $\mathbb{R}/\mathbb{Q}$: that is $\mathbb{R}$ quotiented by the equivalence realtion $x \sim y$ if $x - y \in \mathbb{Q}$. What are the open sets in this space?