Topology
Exercise sheet 5

There are many questions: you do not need to do them all but you should think whether you could do them. Ideally, if I asked you in the tutorial how to do a question you would be able to answer it. They are in no particular order so if you can't do one go on to the next.

1. Suppose that $A \subset \mathbb{R}$ is not compact. Prove that there is a continuous function $f\colon A \to \mathbb{R}$ that is not bounded.

2. Suppose that X is a compact Hausdorff topological space and that A and B are closed subsets of X. Prove that there exist disjoint open sets U and V with $U \supset A$ and $V \supset B$.

   Hint: We showed in lectures that for any point $x \notin A$ we can find sets U_x and V_x with $U_x \supset A$ and $V_x \ni x$. Try and apply the same argument a second time.

3. Suppose that X is a compact topological space and that $(x_n)_{n=1}^{\infty}$ is a sequence of distinct points in A. Prove that the set $\{x_n : n \in \mathbb{N}\}$ has a limit point.

4. Suppose X is a Hausdorff topological space and that $\alpha\colon [0,1] \to X$ is a path.
   (a) Suppose that α is injective. Prove that the image of α (i.e., $\alpha([0,1])$) is homeomorphic to $[0,1]$.
   (b) Prove that this is not true if we do not require α to be injective. Note it is *not* enough just to show that α is not a homeomorphism.
   (c) Give an example where α is not injective but the image of α is homeomorphic to $[0,1]$.

5. Let X be the space of all bounded infinite real-valued sequences with the metric $d(x,y) = \sup_i |x_i - y_i|$. (You might have seen this space called l_∞.) Let 0 denote the all zero sequence. Find a sequence of points in $\overline{B(0,1)}$ (i.e., the closed unit ball of points with $d(x,0) \leq 1$) with no convergent subsequence. Deduce that this closed ball is closed bounded but not compact.

6. Suppose that X is a set and $\mathcal{F}_1$ and $\mathcal{F}_2$ are both topologies on X. We say that $\mathcal{F}_2$ is weaker than $\mathcal{F}_1$ if $\mathcal{F}_2 \subseteq \mathcal{F}_1$ (i.e., every set that is open in the $\mathcal{F}_2$ topology is open in the $\mathcal{F}_1$ topology.)
   (a) Prove that the identity map $(X, \mathcal{F}_1) \to (X, \mathcal{F}_2)$ is continuous.
   (b) Suppose $X = [0,1]$ and $\mathcal{F}_1$ is the usual metric topology and that $\mathcal{F}_2$ is a weaker topology that is Hausdorff. Prove that the two topologies are the same.

7. Prove that the following is a (weird) topology on $\mathbb{N}$. The open sets are $\emptyset, \mathbb{N}$ together with all sets of the form $U_n = \{1, 2, 3, \ldots, n\}$.
   (a) Is $\mathbb{N}$ compact in this topology?
   (b) What are the continuous functions from $\mathbb{N}$ (in this topology) to $\mathbb{R}$? Is there any unbounded continuous function?

8. Here is an alternative proof of the Heine-Borel Theorem that $[0,1]$ is compact. Suppose that U_i $i \in I$ is an open cover with no finite subcover.
   (a) Show that either $[0, 1/2]$ or $[1/2, 1]$ (or both) is not contained in a a finite union of the U_i.
   (b) Repeat the argument on this sub-interval to find an interval of length 1/4 which is not contained in a finite union of the U_i.
   (c) Deduce that there is an increasing sequence a_n and decreasing sequence b_n such that: $a_0 = 0$, $b_0 = 1$, $b_n - a_n = 1/2^n$ and the interval $[a_n, b_n]$ is not contained in a finite union of the U_i.
   (d) Show that the sequence a_n converges to a limit. Call this limit a. Does the sequence b_n converge? If so, what does it converge to?
   (e) The limit a is contained in (at least) one of the U_i. Deduce that this U_i contains $[a_n, b_n]$ for some n and obtain a contradiction.