Topology
Exercise sheet 3

1. Suppose that $X = \mathbb{R}$ with the usual metric topology. We denote by $\mathbb{Q}$ the set of all rational numbers. Find the set of all limit points for the following sets A:

   (a) $A = \mathbb{Q}$

   (b) $A = \mathbb{R} \setminus \mathbb{Q}$

   (c) $A = (-1, 1) \cup (1, 1)$

   (d) $A = \{x_n : n \in \mathbb{N}\}$ where x_n is any convergent sequence.

   (e) (Ill-defined and harder) What are the possibilities if x_n is an arbitrary sequence?

2. Suppose X is a topological space. Prove that any intersection of closed sets is closed, and that any finite union of closed sets is closed. Give an example to show that infinite unions of closed sets need not be closed.

3. Suppose X is an uncountable set with the co-countable topology. Fix x. Prove that for any sequence $(x_n)_{n=1}^{\infty}$ converging to x there is an N such that $\forall n > N : x_n = x$. Deduce that every convergent sequence of points in $X \setminus \{x\}$ converges to something $X \setminus \{x\}$. Is $X \setminus \{x\}$ closed?

4. Find two subsets S, T of $\mathbb{R}$ such that S is homeomorphic to a subset of T and T is homeomorphic to a subset of S, but S and T are not homeomorphic.

5. Suppose X is a topological space and S is a subspace (with the subspace topology). Prove that a subset A of S is closed in S if and only if it is the intersection of a closed (in X) subset B with S.

6. Consider the subset $A = (0, 1]$ in $\mathbb{R}$ (so A is not closed nor is it open). In which of the following subspaces is it open? and in which is it closed.

   (a) $(0, 1]$

   (b) $[0, 1]$

   (c) $(0, \infty)$

   (d) $(-3, 3)$

7. Prove that path connectedness is a topological property. (We shall prove this in the next lecture but it is a good exercise.) You need the following definitions:

   Definition. Suppose X is a topological space. Then a <u>path in X</u> is a continuous function $\alpha\colon [0, 1] \to X$. It <u>starts</u> at $\alpha(0)$ and <u>finishes</u> at $\alpha(1)$.

   Definition. A topological space X is <u>path connected</u> if for any two points $x, y \in X$ there is a path starting at x and finishing at y (abbreviated to 'a path from x to y').