Topology

Exercise sheet 10

1. Find lifts (in $P(\mathbb{R})$) of the following loops in $P_1(S^1, 1)$:
   (a) $e^{2\pi it}$.
   (b) $e^{-2\pi it}$.
   (c) $e^{20\pi it}$.
   (d) The join of the loops in (a) and (b).
   (e) The join of the loops in (c) and (c).

2. For this question assume the result that we just proved: that $\pi_1(S^1, 1)$ is isomorphic to Z.
   (a) For each of the loops in Question 1 say which element of Z they map to under the standard lifting.
   (b) Consider the map $f\colon S^1 \to S^1$ given by $f(x) = x^2$ (thinking in $\mathbb{C}$). Work out what the loops in Question 1 parts (a-c) map to under this function and thus find the induced map $f_*\colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$. (Think of this as a map $\mathbb{Z} \to \mathbb{Z}$ using the isomorphism between $\pi_1(S^1, 1)$ and $\mathbb{Z}$.)
   (c) Do the same for the functions $g, h\colon S^1 \to S^1$ given by $g(z) = 1/z$ and $h(z) = 1$ (i.e., h is constant 1).

3. For this question read and understand the proof of the Brouwer theorem.
   (a) Is there a continuous map $f\colon B^2 \to S^1$ which satisfies $f(x) = x^2$ for all $x \in S^1 \subset B^2$.
   (b) Is there a surjective map from B^2 to $[0, 1]$? Is there a surjective map from B^2 to S^1. If there is such a map f what is the loop $f \circ \alpha$, where $\alpha(t) = e^{2\pi it}$? What is it homotopic to?
   (c) Is there a map $f\colon \mathbb{R} \to \mathbb{R}$ with no fixed point? Is there a map S^1 to S^1 with no fixed point? What about from $[0, 1]$ to $[0, 1]$?

4. Suppose that A, B, C are three closed sets in S^2 with $A \cup B \cup C = S^2$. Prove that there exists $x \in S^2$ such that one of A, B, C contains both x and $-x$.
   Hint. Consider a function $f\colon S^2 \to \mathbb{R}^2$ defined by $f(x) = (d(x, A), d(x, B))$ where d is the (any of the standard) metric on S^2. Use the Borsuk-Ulam theorem.

Second Section

5. Prove that homotopy equivalent spaces have the same fundamental group.

6. Suppose that $f\colon S^1 \to \mathbb{R}$ is a continuous function. Prove that there exists $x \in S^1$ with $f(x) = f(-x)$.

7. Suppose that X, Y are path connected. Prove that $X \times Y$ is path connected. Prove that the fundamental group of $X \times Y$ is $\pi_1(X) \times \pi_1(Y)$.