Probability III – 2008/09

Exercise Sheet 7

Write your name and student number at the top of your assignment before handing it in. Staple all pages together. Return the assignment by 18:30 on Monday, 23 March

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In lectures, we proved several statements about the Birth Process (BP). They can be briefly summarized as follows.

Theorem 1. *Suppose that $X(t)$ is a birth process with $X(0) = 0$. Set $p_n(t) \stackrel{\text{def}}{=} P\{X(t) = n|X(0) = 0\}$ Then the functions $p_n(t)$ satisfy the following equations:*

$$\begin{cases} p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1}p_{n-1}(t) & n \geq 0 \\ p_0(0) = 1 & \\ p_n(0) = 0 & \text{if } \ n > 0. \end{cases} \tag{1}$$

Theorem 2. *Equations (1) have a unique solution which can be found recursively using the following formulae:*

$$\begin{cases} p_0(t) = e^{-\lambda_0 t} \\ p_n(t) = \lambda_{n-1}e^{-\lambda_n t} \int_0^t e^{\lambda_n s} p_{n-1}(s)ds \quad n > 0 \end{cases} \tag{2}$$

Marks

[60] 1. Let $X(t)$ be a Birth Process with parameters $\lambda_0, \lambda_1, ..., \lambda_n, ...$ and suppose that $X(0) = 3$. Set $p_n(t) \stackrel{\text{def}}{=} P\{X(t) = n|X(0) = 3\}$

(a) Using the method explained in lectures derive equations for $p_3(t)$ and for $p_4(t)$.

(b) State the equation for $p_n(t)$.

(c) Derive formulae similar to (2) for $p_3(t)$ and for $p_n(t),\ n \geq 4$.

(d) Suppose that $\lambda_3 = 1$, $\lambda_4 = 1.5$. Find the expressions for $p_3(t)$ and for $p_4(t)$.

(e) Find the probability density function of the time W_4 the process $X(t)$ remains in the state 3.

(f) Hence find mean time $E(W_4|X(0) = 3)$ the process $X(t)$ spends in the state 3.

[40] 2. Consider the birth process (which we briefly discussed in lectures) with $\lambda_n = n\lambda$ $(\lambda > 0)$.

(a) Prove that equations (1) imply that $P\{X(t) = 0|X(0) = 0\} = 1$.

(b) If, however, $X(0) = 1$ then

$$p_n(t) \stackrel{\text{def}}{=} P\{X(t) = n | X(0) = 1\} = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}.$$

Check that this is true.

Hint: since you are given the explicit expressions for $p_n(t)$, it suffices to show that these functions satisfy equations (1) (and relevant initial conditions).

(c) The statement made in (b) means that the random variable $X(t)$ conditioned on $X(0) = 1$ has a geometric distribution. Hence, find $E(X(t)|X(0) = 1)$. Does $E(X(t)|X(0) = 1)$ grow exponentially fast as a function of t?