Probability III – 2008/09

Solutions to Exercise Sheet 6

1. (a)
$$P(X(1)=2) = P(X(1)-X(0)=2) = e^{-\lambda t}\frac{(\lambda t)^2}{2!}\Big|_{\lambda=2,t=1} = 2e^{-2} = 0.2707$$
by (iii) and (ii) in the definition of the Poisson process.

(b) By the definition of the Poisson process, its increments are independent random variables. Hence
$$P(X(1)=2, X(3)=6) = P(X(1)=2, X(3)-X(1)=4) = P(X(1)=2)\,P(X(3)-X(1)=4)$$
and therefore
$$P(X(1)=2, X(3)=6) = e^{-\lambda}\frac{\lambda^2}{2!}e^{-\lambda(3-1)}\frac{(\lambda(3-1))^4}{4!} = \frac{64}{3}e^{-6} = 0.05288,$$

(c) By the theorem proved in lectures, $X(1)$ conditioned on $X(3)=6$ is a Binomial random variable with parameters $(6, \frac{1}{3})$. Hence
$$P(X(1)=2|X(3)=6) = \binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 = \frac{80}{243} = 0.3292$$

(d) Since $P(X(1)=2, X(3)=6) = P(X(1)=2)\times P(X(3)-X(1)=4)$, we have
$$P(X(3)=6|X(1)=2) = \frac{P(X(1)=2, X(3)=6)}{P(X(1)=2)} = P(X(3)-X(1)=4) = \frac{32e^{-4}}{3} = 0.1953$$

2. We must find $P(X(t_1)=k, X(t_2)=n)$ for all values $k, n = 0, 1, 2\ldots$.
$$P(X(t_1)=k, X(t_2)=n) = 0 \quad \text{for all } k > n.$$
This is because $t_1 < t_2$ ($X(t)$ counts events that happened in the interval $(0,t]$, hence $X(t)$ cannot decrease with t).

When $n \geq k$
$$P\{X(t_1)=k,\, X(t_2)=n\} = P\{X(t_1)=k\}\times P\{X(t_2)-X(t_1)=n-k\} = \frac{t_1^k(t_2-t_1)^{n-k}\lambda^n e^{-\lambda t_2}}{k!(n-k)!}.$$

3. If $X(h) \sim \text{Poisson}(\lambda h)$, then

(a)

$$P(X(h)=1) = \lambda h e^{-\lambda h} = \lambda h + \lambda h(e^{-\lambda h}-1) = \lambda h + o(h).$$

The latter equality holds because of

$$\lim_{h\to 0}\frac{\lambda h(e^{-\lambda h}-1)}{h} = \lambda \lim_{h\to 0}(e^{-\lambda h}-1) = 0.$$

(b)

$$P(X(h)=0) = e^{-\lambda h} = 1-\lambda h + (e^{-\lambda h}-1+\lambda h) = 1-\lambda h + o(h)$$

Indeed, by the L'Hopital rule (Calculus I):

$$\lim_{h\to 0}\frac{e^{-\lambda h}-1+\lambda h}{h} = \lim_{h\to 0}\frac{-\lambda e^{-\lambda h}+\lambda}{1} = \lambda \lim_{h\to 0}(1-e^{-\lambda h}) = 0$$

and hence $e^{-\lambda h}-1+\lambda h = o(h)$.

(c)

$$\begin{aligned} P(X(h)\geq 2) &= 1-P(X(h)<2) \\ &= 1-P(X(h)=0)-P(X(h)=1) \\ &= 1-[1-\lambda h+o(h)]-\lambda h-o(h) \\ &= o(h). \end{aligned}$$

4. If 1 hour is one unit of time, then 30 minutes is half a unit.

   $A =$ "one customer entered the store in $0 < t \leq 1$"
   $W_1 =$ the time (in hours) till the first customer enters the store.

   The probability in question is the conditional probability $P(W_1 \leq \frac{1}{2} \mid A)$. Notice that

$$P\{W_1 \leq \frac{1}{2} \mid A\} = P\{X(0.5)=1 \,|\, X(1)=1\} = \frac{1}{2}$$

   because this conditional r. v. has a Binomial distribution with parameters $(1, 0.5)$. (Since $n=1$, it is in fact a Bernoulli distribution.)

5. (a) P.d.f. for W_1 conditioned by the event $X(t)=n$.

   Obviously $0 < W_1 < t$ given that n events occur in $(0,t]$. Therefore, the conditional p.d.f. for W_1 vanishes outside the interval $[0,t]$. If $0 < y < t$, then

$$P\{W_1 > y \,|\, X(t)=n\} = P\{X(y)=0 \,|\, X(t)=n\} = (1-\frac{y}{t})^n.$$

   Hence the p.d.f. for W_1 conditioned by the event $X(t)=n$ is given by

$$f_{W_1|(X(t)=n)}(y) = -\frac{d(1-\frac{y}{t})^n}{dy} = \frac{n}{t}(1-\frac{y}{t})^{n-1}.$$

(b)

$$
\begin{aligned}
E(W_1|X(t)=n) &= \int_0^t w\ f_{W_1|(X(t)=n)}(w)\ dw = \int_0^t n\ \frac{w}{t}\ \left(1-\frac{w}{t}\right)^{n-1}\ dw \\
&= tn \int_0^1 x(1-x)^{n-1}\ dx.
\end{aligned}
$$

and integrating by parts

$$
\begin{aligned}
E(W_1|X(t)=n) &= -t\int_0^1 x\ d(1-x)^{n-1} \\
&= -tx(1-x)^{n-1}\Big|_{x=0}^{x=1} + t\int_0^1 (1-x)^n\ dx \\
&= \frac{t}{n+1}.
\end{aligned}
$$

(c) Limiting (conditional) distribution for W_1 in the limit when $t \to \infty$ and $n = \beta t$.

$$
\begin{aligned}
f_{W_1|(X(t)=n)}(w) &= \frac{n}{t}\left(1-\frac{w}{t}\right)^{n-1} \\
&= \beta\left(1-\frac{\beta w}{n}\right)^{n-1} \qquad [\text{since}\ \ t = \frac{n}{\beta}] \\
&\to \beta e^{-\beta w}, \quad \text{when } n \to \infty.
\end{aligned}
$$

Hence the limiting distribution is exponential with parameter β.

6. $F(W_1, W_2, W_3, W_4, W_5) = W_1 + W_2 + W_3 + W_4 + W_5$ is a symmetric function of its arguments. By the Theorem stated in the Hint,

$$
\begin{aligned}
E(W_1 + W_2 + W_3 + W_4 + W_5|X(1) = 5) &= E(U_1 + U_2 + U_3 + U_4 + U_5) \\
&= 5 \times E(\text{Uniform}([0,1])) = \frac{5}{2},
\end{aligned}
$$

where the random variables U_k, $k = 1, \ldots, 5$, are independent and uniformly distributed on [0,1].