Probability III – 2008/09

Exercise Sheet 6

Write your name and student number at the top of your assignment before handing it in. Staple all pages together. Return the assignment by 18:30 on Monday, 16 March

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To solve problems 1, 2, and 3, you should first recall the axiomatic definition of the Poisson process.

[15] 1. Suppose that customers arrive at a facility according to a Poisson process of rate $\lambda = 2$ customers per hour. Let $X(t)$ be the number of customers that have arrived up to time t. Determine the following probabilities and conditional probabilities:

(a) $P(X(1) = 2)$

(b) $P(X(1) = 2, X(3) = 6)$

(c) $P(X(1) = 2 | X(3) = 6)$

(d) $P(X(3) = 6 | X(1) = 2)$

Give details of your calculations.

[15] 2. For a Poisson process $X(t)$ of rate λ and two fixed times t_1 and t_2, $t_1 < t_2$, find the joint probability mass function for the two random variables $X(t_1)$ and $X(t_2)$.

Hint. Recall the definition of a joint probability mass function of two random variables.

[15] 3. For each value of $h > 0$, let $X(h)$ have a Poisson distribution with parameter λh. Let $p_k = P(X(h) = k)$ for $k = 0, 1, 2, \ldots$. Verify that $p_0 = 1 - \lambda h + o(h)$, $p_1 = \lambda h + o(h)$ and $P(X(h) \geq 2) = o(h)$ as $h \downarrow 0$.

[15] 4. Customers enter a store according to a Poisson process of rate 6 customers per hour. Given that a single customer entered the store during the first hour, what is the probability that this customer entered the store during the first 30 minutes?

[30] 5. Let $X(t)$ be a Poisson process of rate $\lambda > 0$ and W_1 be the time when the first event occurs. Obtain the probability density function for W_1 conditioned by the event "$X(t) = n$". Given that exactly n events occur in the time interval $(0, t]$, calculate for how long, on average, one waits till the first event occurs, i.e. calculate $E(W_1 | X(t) = n)$. Determine the limiting distribution of W_1, under the condition $X(t) = n$ as $n \to \infty$ and $t \to \infty$ in such a way that $n/t = \beta > 0$.

[10] 6. Customers arrive at a certain facility according to a Poisson process of rate $\lambda > 0$. Suppose that it is known that five customers arrived in the first hour. Determine the mean total waiting time $E(W_1 + W_2 + W_3 + W_4 + W_5 | X(1) = 5)$.

Hint. You are supposed to make use of the following statement:

Theorem. *Suppose that* $W_1, W_2, \ldots W_n$ *are occurrence times of a Poisson process of rate* $\lambda > 0$. *Let* $U_1, U_2, \ldots, U_n$ *be a sequence of independent random variables which are uniformly distributed on* $[0,\, t]$. *Let* $R(W_1, \ldots, W_n)$ *be a symmetric function of* n *variables. Then*

$$E\left[R(W_1, \ldots, W_n) | X(t) = n\right] = E\left[R(U_1, \ldots, U_n)\right].$$