The Uniform Distribution and the Poisson Process

1 Definitions and main statements

Let $X(t)$ be a Poisson process of rate $\lambda$. Let $W_1, W_2, ..., W_n$ be the event (the occurrence, or the waiting) times.

**Question:** What is the joint distribution of $W_1, W_2, ..., W_n$ conditioned on the event $X(t) = n$.

It turns out that to answer this question it is convenient to introduce a sequence $U_1, U_2, ..., U_n$ of independent random variables which are uniformly distributed on $[0, T]$. Remember that this by definition means that the joint p.d.f of $U_1, U_2, ..., U_n$ is

$$f_{U_1, ..., U_n}(x_1, x_2, ..., x_n) = f_{U_1}(x_1) \cdot f_{U_2}(x_2) \cdot \ldots \cdot f_{U_n}(x_n) = \begin{cases} \frac{1}{t^n} & \text{if } 0 \leq x_i \leq t \text{ for all } i, \ 1 \leq i \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

where $f_{U_j}(x) = \begin{cases} \frac{1}{t} & \text{if } 0 \leq x \leq t \\ 0 & \text{otherwise} \end{cases}$. Once a sequence of random variables $U_1, U_2, ..., U_n$ is given, we can re-arrange them into a growing sequence $\tilde{W}_1 < \tilde{W}_2 < ... < \tilde{W}_n$ thus obtaining a new sequence of random variables. The remarkable fact is that

The joint distribution of $W_1, W_2, ..., W_n$ conditioned on the event $X(t) = n$ coincides with the joint distribution of $\tilde{W}_1, \tilde{W}_2, ..., \tilde{W}_n$.

It should be appreciated that this statement allows one to replace the conditional distribution of $W_1, W_2, ..., W_n$ by a distribution of a relatively simple sequence of random variables $\tilde{W}_1, \tilde{W}_2, ..., \tilde{W}_n$.

The following theorem is in fact a quantitative version of the above statement:

**Theorem 1.1**

The joint probability density function of $W_1, W_2, ..., W_n$ conditioned on the event $X(t) = n$ is given by

$$f_{\{W_1, ..., W_n \mid X(t) = n\}}(x_1, x_2, ..., x_n) = \begin{cases} \frac{n!}{t^n} & \text{if } 0 \leq x_1 \leq x_2 \leq ... \leq x_n \leq t, \\ 0 & \text{otherwise.} \end{cases}$$

(1)

**Remarks and exercises.** 1. In the case $n = 1$ Theorem 1.1 states that the event time $W_1$ is uniformly distributed on $[0, t]$ given that the total number of events observed by time $t$ is 1. **Exercise:** Prove Theorem 1.1 for $n = 1$ (this is very easy to do).

2. If $n = 2$, formula (1) defines a uniform distribution on a triangle in the $(x_1, x_2)$-plane. **Exercise:** Draw the picture of this triangle. Prove Theorem 1.1 for $n = 2$ (this is slightly more difficult than when $n = 1$).

The random variables $U_j$ are particularly helpful when one wants to find the expectation of $R(W_1, ..., W_n)$, where $R(\cdot)$ is a symmetric function.
Remember that \( R(x_1, \ldots, x_n) \) is said to be a symmetric function if \( R(x_1, \ldots, x_n) = R(x_{i_1}, \ldots, x_{i_n}) \) for any permutation \( (i_1, \ldots, i_n) \) of the sequence \( (1, \ldots, n) \).

**Theorem 1.2**
Suppose that \( W_1, W_2, \ldots, W_n \) are occurrence times of a Poisson process of rate \( \lambda > 0 \). Let \( U_1, U_2, \ldots, U_n \) be a sequence of independent random variables which are uniformly distributed on \([0, t]\). Let \( R(W_1, \ldots, W_n) \) be a symmetric function of \( n \) variables. Then
\[
E(R(W_1, \ldots, W_n)|X(t) = n) = E[R(U_1, \ldots, U_n)].
\] (2)

### 2 Extended example

Customers arrive at a facility as a Poisson Process of rate \( \lambda \). There is a waiting cost of \( \£C \) per person per unit of time. Customers gather at the facility and are dispatched at time \( T \) irrespective of the number of customers. Each dispatch costs \( \£k \) (irrespective of the number of customers).

**Question:** How should \( T \) be chosen to minimize the expected cost per unit of time?

**Solution.** First of all
\[
\text{dispatching cost per unit of time} = \frac{k}{T}.
\]

Next
\[
E(\text{waiting cost per unit of time}) = \frac{1}{T} \sum_{n=0}^{\infty} E[C(T - W_1) + C(T - W_2) + \ldots C(T - W_n)|X(t) = n] P\{X(T) = n\}.
\]

By Theorem 1.2,
\[
E[C(T - W_1) + C(T - W_2) + \ldots C(T - W_n)|X(t) = n] = CE\left[nT - \sum_{i=1}^{n} U_i \right] = Cn \frac{T}{2}.
\]

Hence
\[
E(\text{waiting cost per unit of time}) = \frac{1}{T} \sum_{n=0}^{\infty} Cn \frac{T}{2} \times P\{X(T) = n\} = C \frac{\sum_{n=0}^{\infty} ne^{-\lambda T} (\lambda T)^n}{\sum_{n=0}^{\infty} n!} = C \frac{\lambda T}{2}.
\]

Thus
\[
E(\text{total cost per unit of time}) = C \frac{\lambda T}{2} + k \frac{k}{T} = f(T).
\]
But then \( f'(T) = C \frac{\lambda T}{2} - \frac{k^2}{2T^2} = 0 \) implies \( T = \sqrt{\frac{2k}{C\lambda}} \).

**Example 2.** Study very carefully Problem 5 from CW6.